A NOTE ON NORMAL DIVISION ALGEBRAS OF PRIME DEGREE*

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Wedderburn has proved† that all normal division algebras of degree three over a non-modular field $\mathfrak{K}$ are cyclic algebras. It is easily verified that his proof is actually correct for $\mathfrak{K}$ of any characteristic not three, and I gave a modification of his proof‡ showing the result also valid for the remaining characteristic three case. Attempts to generalize Wedderburn's proof to algebras of prime degree $p>3$ have thus far been futile, and it is not yet known whether there are any non-cyclic algebras of prime degree. One notes that in both Wedderburn's proof and my modification one starts by studying a non-cyclic cubic field and thus a subfield of a normal splitting field of degree six with a quadratic (cyclic) subfield. I have generalized this property to the case of arbitrary prime degree and have now provided a new proof of the Weddenburn theorem for algebras of degree three in the characteristic three case. The result is the special case $p=3$, $m=2$ of the following theorem:

THEOREM. *Let $\mathfrak{D}$ be a normal division algebra of degree p over a field $\mathfrak{K}$ of characteristic p, and let m be prime to p. Then if $\mathfrak{D}$ has a normal splitting field $\mathfrak{W}$ of degree pm over $\mathfrak{K}$, with a cyclic subfield $\mathfrak{L}$ of degree m over $\mathfrak{K}$, it follows that the algebra $\mathfrak{D}$ is a cyclic algebra.*

In our proof we shall use the following known theorems§ on normal division algebras $\mathfrak{D}$ of degree n over arbitrary fields $\mathfrak{K}$:

LEMMA 1. *Let $\mathfrak{L}$ have degree prime to n. Then $\mathfrak{D}_{\mathfrak{L}}$ is a division algebra.*

LEMMA 2. *Let $\mathfrak{Z}_0$ have degree n over $\mathfrak{K}$ and split $\mathfrak{D}$. Then $\mathfrak{Z}_0$ is equivalent to a (maximal) subfield of $\mathfrak{D}$.*

LEMMA 3. *Let $\mathfrak{D}$ have a cyclic subfield of degree n. Then $\mathfrak{D}$ is a cyclic algebra.*

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† Transactions of this Society, vol. 22 (1921), pp. 129–135.

‡ Transactions of this Society, vol. 36 (1934), pp. 388–394.

§ Cf. Deuring's *Algebren* for our notation and the proofs of the results of Lemmas 1, 2, 3. Lemma 4 was proved by the author for $\mathfrak{K}$ of characteristic not p, Transactions of this Society, vol. 36 (1934), pp. 885–892, and for $\mathfrak{K}$ of characteristic p, ibid., vol. 39 (1936), pp. 183–188.

LEMMA 4. *Let $\mathfrak{D}$ of prime degree $n=p$ over $\mathfrak{K}$ have a splitting field $\mathfrak{Y}=\mathfrak{K}(y)$, such that $y^p=\gamma$ in $\mathfrak{K}$. Then $\mathfrak{D}$ is a cyclic algebra.*

To make our proof we let $\mathfrak{G}$ be the automorphism group of $\mathfrak{W}$ over $\mathfrak{K}$ and $\mathfrak{H}$ the subgroup of $\mathfrak{G}$ corresponding to $\mathfrak{L}$. Then $\mathfrak{H}$ is a normal divisor of $\mathfrak{G}$ and is of prime order p; $\mathfrak{H}=[S]$ is a cyclic group. The group of the cyclic field $\mathfrak{L}$ over $\mathfrak{K}$ is the quotient group $\mathfrak{G}/\mathfrak{H}$ and is a cyclic group $[\mathfrak{H}T]$. Here T is an automorphism of $\mathfrak{G}$ and $T^m=S^a$ in $\mathfrak{H}$. But then $[\mathfrak{H}T^p]=[\mathfrak{H}T]$ since p is prime to m, $(\mathfrak{H}T)^p=\mathfrak{H}T^p$, and $T^{pm}=S^{pa}=I$. Hence we may assume without loss of generality that $T^m=I$. Since $\mathfrak{H}T$ has order m so does T. The cyclic subgroup $\mathfrak{T}=[T]$ of $\mathfrak{G}$ corresponds to a subfield $\mathfrak{Z}_0$ of degree p over $\mathfrak{K}$ of $\mathfrak{W}$, and we have the following lemma:

LEMMA 5. *The field $\mathfrak{Z}_0$ splits $\mathfrak{D}$.*

For clearly $\mathfrak{W}$ is the composite of $\mathfrak{Z}_0$ and $\mathfrak{L}$, and $\mathfrak{W}=(\mathfrak{Z}_0)_{\mathfrak{L}}$. Now $\mathfrak{D}$ has prime degree, and either $\mathfrak{Z}_0$ splits $\mathfrak{D}$ or $\mathfrak{D}_{\mathfrak{Z}_0}$ is a division algebra. In the latter case by Lemma 1 the algebra $(\mathfrak{D}_{\mathfrak{Z}_0})_{\mathfrak{L}}=\mathfrak{D}_{\mathfrak{W}}$ is a division algebra, contrary to our hypothesis that $\mathfrak{W}$ splits $\mathfrak{D}$.

Since $\mathfrak{H}$ is a normal divisor of $\mathfrak{G}$ we have $T\mathfrak{H}=\mathfrak{H}T$, $TS=S^eT$. If $e=1$, then the group $[T]$ is a normal divisor of $\mathfrak{G}$, and $\mathfrak{Z}_0$ is cyclic of degree p over $\mathfrak{K}$. By Lemmas 5 and 3 the algebra $\mathfrak{D}$ is cyclic. There remains the case $e>1$.

Now $T^2S=TS^eT=S^{e^2}T$, $\cdots$, $T^mS=S^{e^m}T^m=S=S^{e^m}$. Since S has order p we have

$$e^m \equiv 1 \pmod{p}, \qquad 0 < e \leqq p-1. \tag{1}$$

We let ν be the least positive integer such that $e^\nu \equiv 1 \pmod{p}$. Now $\nu \neq 1$, and ν must divide both $p-1$ and m. It follows that

$$m = \nu q, \qquad p-1 = \mu\nu \tag{2}$$

for integers μ and q. Notice that the group $[T]$ is not a normal divisor of $\mathfrak{G}$, so that $\mathfrak{Z}_0$ *is not a cyclic field over* $\mathfrak{K}$.

By Lemmas 2, 5 the algebra $\mathfrak{D}$ has a subfield $\mathfrak{Z}$ of degree p over $\mathfrak{K}$ equivalent to $\mathfrak{Z}_0$. Evidently $\mathfrak{Z}_{\mathfrak{L}}$ is equivalent to $\mathfrak{W}$, and $\mathfrak{Z}_{\mathfrak{L}}=\mathfrak{Z}\times\mathfrak{L}$. But the group of $\mathfrak{W}$ over $\mathfrak{L}$ is $\mathfrak{H}$; $\mathfrak{Z}_{\mathfrak{L}}$ is cyclic of degree p over $\mathfrak{L}$ with generating automorphism which we shall designate by S. Moreover if z is in $\mathfrak{Z}_{\mathfrak{L}}$, the automorphism S which is given by $z \longleftrightarrow z^S$ goes into $z^T \longleftrightarrow (z^S)^T=(z^T)^{S^e}$ which is the automorphism S^e of $\mathfrak{Z}_{\mathfrak{L}}$.

By Lemma 3 we have $\mathfrak{D}_{\mathfrak{L}}=\mathfrak{D}\times\mathfrak{L}=(\mathfrak{Z}_{\mathfrak{L}}, S, g)$ for g in $\mathfrak{L}$. This algebra has the automorphism

$$d \longleftrightarrow d, \quad \lambda \longleftrightarrow \lambda^T, \qquad d \text{ in } \mathfrak{D}, \ \lambda \text{ in } \mathfrak{L}. \tag{3}$$

Apply this automorphism to $\mathfrak{D}\times\mathfrak{P}$ and obtain

$$\mathfrak{D}\times\mathfrak{P} = (\mathfrak{Z}_{\mathfrak{P}}, S^{e}, g^{T}). \tag{4}$$

But then it is known that

$$\mathfrak{D} = (\mathfrak{Z}^{\mathfrak{P}}, S, (g^{T})^{f}) \sim (\mathfrak{Z}_{\mathfrak{P}}, S, g^{T})^{f}, \tag{5}$$

where f is chosen so that $ef\equiv 1 \pmod{p}$. It follows that

$$\mathfrak{D} \sim (\mathfrak{Z}, S, g^{T^{j}})^{f^{j}}, \qquad j = 1, 2, \cdots, n. \tag{6}$$

We form $g_0 = gg^{T^{\nu}}\cdots g^{T^{\nu(q-1)}}$ which is in the cyclic subfield Λ of $\mathfrak{P}$ of degree ν over $\mathfrak{K}$. Now

$$\mathfrak{A} = (\mathfrak{Z}_{\mathfrak{P}}, S, g)\times(\mathfrak{Z}_{\mathfrak{P}}, S, g^{T^{\nu}})\times\cdots\times(\mathfrak{Z}_{\mathfrak{P}}, S, g^{T^{\nu(q-1)}}) \sim (\mathfrak{Z}_{\mathfrak{P}}, S, g_0) \tag{7}$$

over $\mathfrak{P}$. But $\mathfrak{A}\sim(\mathfrak{D}_{\mathfrak{P}})^{\alpha}$, where by (6) we have

$$\alpha = 1 + f^{\nu} + f^{2\nu} + \cdots + f^{(q-1)\nu} \equiv q \pmod{p}, \tag{8}$$

since $e^{\nu}\equiv 1 \pmod{p}$, $ef\equiv 1 \pmod{p}$, $(ef)^{\nu}\equiv f^{\nu}\equiv 1 \pmod{p}$. Now q is prime to p; hence $qq_0\equiv 1 \pmod{p}$, and $\mathfrak{A}^{q_0}\sim(\mathfrak{Z}_{\mathfrak{P}}, S, g_0{}^{q_0})\sim(\mathfrak{D}^{qq_0})_{\mathfrak{P}}\sim\mathfrak{D}_{\mathfrak{P}}$, where $g_0{}^{q_0}$ is in Λ. It follows that there is no loss of generality if we assume that g is in Λ. We shall make this assumption.

By (6) we have

$$(\mathfrak{D}_{\mathfrak{P}})^{r} \sim (\mathfrak{Z}_{\mathfrak{P}}, S, g)\times(\mathfrak{Z}^{\mathfrak{P}}, S, g^{T})^{f}\times\cdots \times(\mathfrak{Z}_{\mathfrak{P}}, S, g^{T^{r-1}})^{f^{r-1}} \sim (\mathfrak{Z}_{\mathfrak{P}}, S, \gamma_0), \tag{9}$$

where

$$\gamma_0 = \prod_{k=1}^{r} (g^{T^{k}})^{f^{k}}. \tag{10}$$

But then

$$\gamma_0{}^{T} = \prod_{k=1}^{r} (g^{T^{k+1}})^{f^{k}}, \quad \gamma_0{}^{e} = \prod_{k=1}^{r} (g^{T^{k}})^{ef^{k}}. \tag{11}$$

Since $ef\equiv 1 \pmod{p}$ we have

$$\gamma_0{}^{T} = \lambda_0{}^{p}\gamma_0{}^{e}, \qquad \lambda_0 \text{ in } \Lambda. \tag{12}$$

Now $\nu\nu_0\equiv 1 \pmod{p}$ and $(\mathfrak{D}_{\mathfrak{P}})^{\nu\nu_0}\sim\mathfrak{D}_{\mathfrak{P}}\sim(\mathfrak{Z}_{\mathfrak{P}}, S, \gamma_1)$, where $\gamma_1=\gamma_0{}^{\nu_0}$, and (12) implies that

$$\gamma_1{}^{T} = \lambda^{p}\gamma_1{}^{e}, \qquad \lambda \text{ in } \Lambda. \tag{13}$$

Since $\mathfrak{D}_{\mathfrak{P}}$ and $(\mathfrak{Z}_{\mathfrak{P}}, S, \gamma_1)$ have the same order, they are equivalent, and we have proved the following lemma:

LEMMA 6. *The algebra* $\mathfrak{D}_{\mathfrak{L}}$ *has the generation* $\mathfrak{D}_{\mathfrak{L}} = (\mathfrak{Z}_{\mathfrak{L}}, S, \gamma)$ *where* γ_1 *is in* Λ *and* (13) *holds.*

The cyclic algebra $\mathfrak{D}_{\mathfrak{L}}$ contains a quantity y_0 such that $y_0^p = \gamma_1$, and $\mathfrak{L}(y_0)$ is a maximal subfield of $\mathfrak{D}_{\mathfrak{L}}$. Hence $\mathfrak{L}(y_1) \cong \mathfrak{L}(y_0)$ is a scalar splitting field of $\mathfrak{D}_{\mathfrak{L}}$. But by (13) we have

$$\gamma_1 T^j = \lambda_j^p \gamma_1 e^j, \qquad j = 0, 1, \cdots, \nu - 1; \tag{14}$$

and if

$$y = y_1 + \lambda_1 y_1^e + \lambda_2 y_1^{e^2} + \cdots + \lambda_{\nu-1} y_1^{e^{\nu-1}}, \tag{15}$$

then $\mathfrak{L}(y_1) = \mathfrak{L}(y)$. For $0 < e \leqq p-1$, $e^i \equiv e^j \pmod{p}$ if and only if $i-j$ is divisible by ν; y is clearly not in $\mathfrak{L}$, and y in $\mathfrak{L}(y_1)$ generates $\mathfrak{L}(y_1)$. It follows that $\mathfrak{L}(y)$ splits $\mathfrak{D}_{\mathfrak{L}}$. But $\mathfrak{K}$ has characteristic p and

$$y^p = \gamma_1 + \gamma_1 T + \cdots + \gamma_1 T^{p-1} = \gamma \text{ in } \mathfrak{K}. \tag{16}$$

Now $\mathfrak{L}(y) = [\mathfrak{K}(y)]_{\mathfrak{L}}$, and $\mathfrak{K}(y)$ splits $\mathfrak{D}$ by the proof of Lemma 5. By Lemma 4, $\mathfrak{D}$ is a cyclic algebra.

In closing let us note that all of our proof is valid for arbitrary fields except the final result (16), which depends essentially* upon the property that $\mathfrak{K}$ has characteristic p.

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* Added in proof: When $p=3$ we may replace (13) by $\gamma_1 T = \gamma_1^{-1}$, and direct computation shows that if a is in $\mathfrak{Z}$ with trace zero and norm α, and $u = a(1+y_1+y_1^{-1})$, then $u^3 = \alpha(2+\gamma_1+\gamma_1 T)$ in $\mathfrak{K}$. This proves $\mathfrak{D}$ cyclic for any characteristic.

QUELQUES REMARQUES SUR LES RELATIONS ENTRE LES NOTIONS D'ÉCART RÉGULIER ET DE DISTANCE

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Madame A. Frink a donné récemment* une nouvelle démonstration très élégante du théorème suivant de M. E. W. Chittenden: *Si dans un espace E la notion de limite est définie par un écart régulier r,*† *elle peut être définie également par une distance ρ.*

La démonstration de Mme Frink repose sur une construction de la distance ρ qui, à cause de sa simplicité, permet d'établir des relations étroites entre les propriétés métriques de l'écart r et de la distance ρ.

Dans ce qui suit, nous allons indiquer des relations de ce genre dans quelques cas simples. Par r nous désignerons un écart régulier défini dans l'espace E et par ρ la distance correspondante construite par Mme Frink.

1. **Les écarts complets.** Nous dirons qu'une suite de points $\{p_n\}$ de E est une *suite de Cauchy selon r*, si $\lim_{i=\infty, j=\infty} r(p_i, p_j)=0$. Si toute suite de Cauchy selon r converge dans E, l'écart r sera dit *complet*.

De la démonstration de Mme. Frink résulte qu'il existe deux fonctions croissantes $\mu_1(\eta)$ et $\mu_2(\eta)$ telles que $\mu_i(\eta)>0$ pour $\eta>0$, $\mu_i(0)=\lim_{\eta=0}\mu_i(\eta)=0$, et que

$$\mu_1(r(p, q)) \leqq \rho(p, q) \leqq \mu_2(r(p, q)), \tag{1}$$

pour tous les p, q de E.‡

Il s'ensuit de (1) immédiatement qu'une suite de Cauchy selon r est en même temps une suite de Cauchy selon la distance ρ et vice versa. De ce fait découle le théorème suivant:

THÉORÈME 1. *Pour que la distance ρ soit complète, il faut et il suffit que l'écart r le soit.*

2. **La presque-convexité.** Un écart r' est dit presque-convexe, si, pour tous a, b de E et tout $\eta>0$, il existe un c de E tel que

* Ce Bulletin, vol. 43 (1937), p. 133.

† On dit d'apres M. Fréchet que dans E est défini un écart régulier r, si pour tout couple de points (p, q) de E est défini un nombre $r(p, q)\geqq 0$ tel que: (1) $r(p, q)=0$ *équivaut à* $p=q$, (2) $r(p, q)=r(q, p)$, et (3) *il existe une fonction positive croissante* $\delta(\epsilon)$, *pour* $\epsilon>0$, *telle que, si* $r(x, y)\leqq\delta(\epsilon)$ *et* $r(y, z)\leqq\delta(\epsilon)$, $r(x, z)\leqq\epsilon$.

‡ Les fonctions $\mu_i(\eta)$ ne dépendent que de la fonction $\delta(\epsilon)$ qui correspond à l'écart r. Pour $\delta(\epsilon)=\epsilon/2$, on peut prendre $\mu_1(\eta)=\eta/4$ et $\mu_2(\eta)=\eta$.

$$(2.1) \qquad |r'(a, c) - \tfrac{1}{2}r'(a, b)| < \eta, \quad |r'(c, b) - \tfrac{1}{2}r'(a, b)| < \eta.^*$$

Cette propriété entraîne évidemment la suivante:

(A) *Pour tous a, b de E, $\eta > 0$, et n entier positif, il existe une suite de points c_k, $(k = 1, 2, \cdots, (2^n - 1))$, tels que, en posant $a = c_0$ et $b = c_{2^n}$, on ait*

$$\left| r'(c_k, c_{k+1}) - \frac{1}{2^n} r'(a, b) \right| < \eta, \qquad k = 0, 1, 2, \cdots, (2^n - 1).$$

Supposons maintenant que l'écart r satisfait à la condition suivante:

(X) *Si $r(x, y) \leqq \epsilon$ et $r(y, z) \leqq \epsilon$, $r(x, z) \leqq 2\epsilon$.*†

THÉORÈME 2. *Si l'écart r, satisfaisant à* (X), *est presque-convexe, la distance ρ l'est également.*

DÉMONSTRATION. Revenons à la définition de la distance ρ. Pour un écart r satisfaisant à (X), elle s'écrit‡

$$(2.2) \qquad \rho(a, b) = \text{borne inf.} \sum_{i=0}^{N-1} r(p_i, p_{i+1}),$$

$\{p_i\}$, $(i = 0, 1, \cdots, N)$, parcourant toutes les suites finies de points de E avec $p_0 = a$ et $p_N = b$.

Pour un $\eta > 0$ quelconque choisissons une telle suite avec

$$(2.3) \qquad \left| \rho(a, b) - \sum_{i=0}^{N-1} r(p_i, p_{i+1}) \right| < \frac{\eta}{4}.$$

Dans la suite $\{p_i\}$ soit p_k le dernier terme pour lequel la somme $\sum_{i=0}^{k-1} r(p_i, p_{i+1})$ est encore $\leqq \frac{1}{2}\rho(a, b)$. D'après (2. 2), $k < N$. On a donc

$$(2.4) \quad 0 \leqq \xi < r(p_k, p_{k+1}), \quad \text{où} \quad \xi = \tfrac{1}{2}\rho(a, b) - \sum_{i=0}^{k-1} r(p_i, p_{i+1}).$$

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* La notion de presque-convexité a un sens pour les écarts les plus généraux, mais elle présente le plus d'intérêt quand r' est une distance. Elle a été introduite pour les distances dans notre thèse *Sur la métrique et la métrisation*, Varsovie, 1930 (non publiée). M. K. Menger a mentionné cette notion dans son article *Bericht über metrische Geometrie*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 40 (1931), p. 207. Il est à remarquer que pour une distance r', la presque-convexité entraîne la propriété plus forte suivante: *pour toute décomposition $r'(a, b) = \alpha + \beta$, avec $\alpha \geqq 0$, $\beta \geqq 0$, et pour tout $\eta > 0$, il existe un point c avec $|r'(a, c) - \alpha| < \eta$ et $|r'(c, b) - \beta| < \eta$.*

† La condition (X) signifie que $\delta(\epsilon) = \epsilon/2$.

‡ Cf. A. Frink, loc. cit., p. 135.

Choisissons alors un n entier positif de sorte que

$$\frac{1}{2^n} r(p_k, p_{k+1}) < \frac{\eta}{4}, \tag{2.5}$$

et considérons la suite $\{q_j\}$, existant d'après (A), telle que

$$q_0 = p_k,\ q_{2^n} = p_{k+1},\ \left| r(q_j, q_{j+1}) - \frac{1}{2^n} r(p_k, p_{k+1}) \right| < \frac{\eta}{4 \cdot 2^n}, \tag{2.6}$$

$$j = 0, 1, \cdots, (2^n - 1).$$

Dans la suite $\{q_j\}$ soit q_h le dernier terme avec $\sum_{j=0}^{h-1} r(q_j, q_{j+1}) \leqq \xi$, donc

$$0 \leqq \xi - \sum_{j=0}^{h-1} r(q_j, q_{j+1}) < r(q_h, q_{h+1}) < \frac{\eta}{4} + \frac{\eta}{4 \cdot 2^n} < \frac{2\eta}{4}.^* \tag{2.7}$$

D'après la définition de la distance ρ, on tire des equations (2.4) et (2.7)

$$\begin{aligned} \rho(a, q_h) &\leqq \rho(a, p_k) + \rho(p_k, q_h) \\ &\leqq \sum_{i=0}^{k-1} r(p_i, p_{i+1}) + \sum_{j=0}^{h-1} r(q_j, q_{j+1}) \leqq \tfrac{1}{2}\rho(a, b). \end{aligned} \tag{2.8}$$

Les inégalités de (2.3) à (2.7) nous donnent ensuite

$$\begin{aligned} \rho(q_h, b) &\leqq \rho(q_h, p_{k+1}) + \rho(p_{k+1}, b) \\ &\leqq \sum_{j=h}^{2^n-1} r(q_j, q_{j+1}) + \sum_{i=k+1}^{N-1} r(p_i, p_{i+1}) \\ &= \left[\sum_{j=0}^{2^n-1} r(q_j, q_{j+1}) - \sum_{j=0}^{h-1} r(q_j, q_{j+1}) \right] \\ &\quad + \left[\sum_{i=0}^{N-1} r(p_i, p_{i+1}) - \sum_{i=0}^{k} r(p_i, p_{i+1}) \right] \\ &< r(p_k, p_{k+1}) + \frac{\eta}{4} - \xi + \frac{2\eta}{4} \\ &\quad + \rho(a, b) + \frac{\eta}{4} + \xi - \frac{1}{2}\rho(a, b) - r(p_k, p_{k+1}) \\ &= \tfrac{1}{2}\rho(a, b) + \eta. \end{aligned} \tag{2.9}$$

* Si $h = 2^n$, les inégalités (2.7) restent vraies à condition qu'on y efface le membre $r(q_h, q_{h+1})$ (ceci se déduit de (2.4) et (2.6).

En posant $c=q_k$ on déduit des inégalités (2.8) et (2.9) immédiatement que

$$|\rho(a, c) - \tfrac{1}{2}\rho(a, b)| < \eta \quad \text{et} \quad |\rho(c, b) - \tfrac{1}{2}\rho(a, b)| < \eta,$$

ce qu'il fallait démontrer.

3. **Remarque sur la notion de distance géodésique.** Le Théorème 2 peut être considéré comme une conséquence d'un théorème concernant la distance géodésique dans un espace métrique général. On peut définir la distance géodésique de différentes manières.* D'un certain point de vue il est avantageux de la définir de manière suivante:

Soit r' un écart soumis uniquement aux conditions suivantes:

$$r'(x, y) \geqq 0, \qquad r'(x, x) = 0. \tag{3.1}$$

On appelle *ϵ-chaîne unissant a et b* une suite $\{p_i\}$, $(i=0, 1, \cdots, N)$, telle que

$$p_0 = a,\ p_N = b,\ r'(p_i, p_{i+1}) \leqq \epsilon, \qquad i = 0, 1, \cdots, N-1. \tag{3.2}$$

On appelle *longueur* de la chaîne $\{p_i\}$, la quantité

$$L\{p_i\} = \sum_{i=0}^{N-1} r'(p_i, p_{i+1}). \tag{3.3}$$

Une chaîne $\{p_i'\}$ est dite *subdivision* de $\{p_i\}$, si elle s'obtient de cette dernière en intercalant entre certains (ou entre tous) éléments consécutifs p_i, p_{i+1} de nouveaux éléments.

Nous dirons qu'une suite des chaînes $\{\{p_i^{(k)}\}\}$, $(k=1, 2, \cdots)$, est *une suite convergente de subdivisions liant a à b*, si (1) $\{p_i^{(k)}\}$ sont des ϵ_k-chaînes unissant a et b, avec $\lim_{k=\infty} \epsilon_k = 0$; (2) $\{p_i^{(k+1)}\}$ est une subdivision de $\{p_i^{(k)}\}$, et (3) $\lim_{k=\infty} L\{p_i^{(k)}\}$ existe (mais peut être fini ou infini).

La "distance" géodésique $\bar{\rho}(a, b)$ peut être maintenant définie comme suit:

$$\bar{\rho}(a, b) = \text{borne inf.} \lim_{k=\infty} L\{p_i^{(k)}\}, \tag{3.4}$$

les $\{\{p_i^{(k)}\}\}$ parcourant toutes les suites convergentes de subdivisions liant a à b.†

* M. K. Menger a considéré, dans les espaces distancés, une distance géodésique égale, d'après le modèle classique, à la borne inférieure des longueurs des arcs unissant deux points a et b de E (cf. Mathematische Annalen, vol. 103 (1930), p. 492).

† Il est à remarquer que si r' est une *distance complète*, $\bar{\rho}$ se confond avec la distance géodésique formée d'après le modèle classique.

On démontre pour $\bar{\rho}$ les propriétés suivantes: (i) $\bar{\rho}(x,y) \geqq 0$, $\bar{\rho}(x, x) = 0$ (mais $\bar{\rho}(x, y)$ peut être infini, $\bar{\rho}(x, y)$ peut être $= 0$ sans que $x = y$ et il se peut que $\bar{\rho}(x, y) \neq \bar{\rho}(y, x)$); (ii) $\bar{\rho}(x, y) + \bar{\rho}(y, z) \geqq \bar{\rho}(x, z)$. A cause de l'inégalité triangulaire (ii) on peut considérer $\bar{\rho}$ comme une sorte de distance.

On démontre de plus la propriété: (iii) *la "distance" géodésique $\bar{\rho}$, si elle est finie, est toujours presque-convexe.*

En revenant maintenant aux conditions du Théorème 2, on prouve d'abord que la distance ρ est égale à la distance géodésique construite à l'aide de l'écart r et le théorème résulte alors de la propriété (iii).

4. **Les écarts complets et presque-convexes.** Les Théorèmes 1 et 2 donnent ensemble le théorème suivant:

THÉORÈME 3. *Si l'écart r, satisfaisant à la condition* (X) *du* §2, *est complet et presque-convexe, il en est de même de la distance ρ.*

L'intérêt de ce théorème réside dans le fait qu'un espace avec une distance complète et presque-convexe possède des propriétés remarquables. Ainsi par exemple, dans un tel espace, deux points quelconques a et b peuvent être liés par un arc *presque-géodésique* de longueur inférieure à $\rho(a, b) + \epsilon$, pour $\epsilon > 0$ quelconque. Il en résulte la propriété topologique suivante: *l'espace est quasi-péanien.**

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* D'après M. K. Borsuk on appelle ainsi un espace connexe, localement connexe, et métrisable à l'aide d'une distance complète.

A SYMMETRIC REPRESENTATION OF THE TWENTY-SEVEN LINES ON A CUBIC SURFACE BY LINES IN A FINITE GEOMETRY*

J. S. FRAME

1. **Introduction.** The group G_{51840} of the automorphisms of the twenty-seven lines and forty-five tritangent planes on a general cubic surface has an even subgroup G_{25920} which is simple. This may be represented on the one hand as the linear group $A(4, 3)$, and on the other hand as the linear group $HO(4, 4)$.† Each of these linear groups suggests a representation of the configuration of the lines on the cubic surface by lines in a finite geometry. Coble‡ has analysed the invariant configurations of the finite projective geometry $PG(3, 3)$§ under the group $A(4, 3)$. In this paper we shall examine those configurations of the $PG(3, 4)$ under the group $HO(4, 4)$ which are isomorphic to the configurations of lines and planes on the general cubic surface.

The notation to be developed in this paper assigns coordinates in a symmetric manner to the twenty-seven lines and forty-five tritangent planes on the cubic surface and affords extremely simple conditions to determine their incidences. For this reason it has some advantages over the commonly used double-six notation devised by Schläfli.‖

2. **The planes.** We assign to the planes of the $PG(3, 4)$ a set of four homogeneous coordinates (u_0, u_1, u_2, u_3) which are marks of the field $F \equiv GF(2^2)$. The four marks of F, which we denote by 0, 1, ω, $\bar{\omega}$, are roots of the congruence

$$u^4 \equiv u \pmod 2. \tag{1}$$

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* Presented to the Society, February 26, 1938.

† L. E. Dickson, *Linear Groups*, 1901, p. 309. J. S. Frame, *The simple group of order* 25920, Duke Mathematical Journal, vol. 2 (1936), p. 477.

‡ A. B. Coble, *A Configuration in Finite Geometry Isomorphic with That of the Twenty-seven Lines of a Cubic Surface*, Johns Hopkins University Circulars, no. 208, 1908, pp. 80–88.

§ R. D. Carmichael discusses finite projective geometries in his *Introduction to the Theory of Groups of Finite Order*, 1937.

‖ L. Schläfli, *On the twenty-seven lines upon a surface of third order*, Quarterly Journal of Mathematics, vol. 2 (1858), pp. 110–120. A. Henderson, *The Twenty-seven Lines upon the Cubic Surface*, 1911. For other notations, see H. S. M. Coxeter, *Polytopes with regular-prismatic vertex figures*, Philosophical Transactions of the Royal Society of London, vol. 229 (1930), pp. 396, 418.

Defining the conjugate $\bar{u}$ of a mark u to be its square, we use the abbreviation

$$\sum_{i=0}^{3} \bar{u}_i v_i = u \cdot v, \tag{2}$$

and we call two planes, (u_0, u_1, u_2, u_3) and (v_0, v_1, v_2, v_3), orthogonal if $u \cdot v = 0$.

The group $HO(4, 4)$ leaves invariant the bilinear form $u \cdot v$. Of the $(4^4-1)/(4-1)=85$ planes u, the forty-five for which $u \cdot u = 0$ are permuted transitively among themselves, and so are the remaining forty for which $u \cdot u = 1$. We shall call the former isotropic, and the latter non-isotropic. Upon inspection we find that each of the forty-five isotropic planes is orthogonal to twelve others, forming three sets of four linearly dependent planes. For example, the plane u: (1, 1, 1, 1) is orthogonal to each of the twelve planes $ku+v$, where $k \in F$, and v is one of the three planes (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0). If u and v are any two mutually orthogonal isotropic planes, just five isotropic planes $ku+lv$ pass through their line of intersection. Each plane contains just three such lines, which form a triangle. Each line meets ten others, which lie by pairs in the five incident isotropic planes. This proves the isomorphism described in the following theorem:

THEOREM 1. *The forty-five tritangent planes to the general cubic surface are in one-to-one correspondence with the forty-five isotropic planes of the finite geometry PG(3, 4), in such manner that two tritangent planes which intersect in a line of the surface correspond to two isotropic planes which are orthogonal.*

3. **The lines.** A simple notation for the line of intersection of two isotropic planes u and v may be derived from the six Pluecker coordinates, $p_{ij} = u_i v_j - u_j v_i$, of the line. These coordinates are connected by the identity

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0. \tag{3}$$

The conditions $u \cdot u = v \cdot v = u \cdot v = 0$ can be shown to imply that both of the factors in one of the terms of equation (3) must vanish. The other four determinants p_{ij} must be different from zero if the planes u and v are distinct. Two suitably chosen ratios of these four quantities, together with the relation (3), suffice to determine a line. In order to preserve symmetry, while avoiding division by zero, we may use the conjugate of a mark of the field F instead of its reciprocal. Accordingly, we assign to a line the three non-homogeneous coordinates

$$(\bar{p}_{01}p_{23}, \bar{p}_{02}p_{31}, \bar{p}_{03}p_{12}), \tag{4}$$

of which just one is zero. For each of the three positions of the zero, there are nine ways of choosing the other two coordinates. These twenty-seven possibilities correspond to the twenty-seven lines on the cubic surface.

In terms of these coordinates, the condition that two lines of the cubic surface intersect is simply that they have just one coordinate in common.

THEOREM 2. *The twenty-seven lines of the general cubic surface are in one-to-one correspondence with the twenty-seven number triples*

$$(5) \qquad (0, b, c), (a, 0, c), (a, b, 0), \qquad a, b, c = 1, \omega, \text{ or } \bar{\omega},$$

in such manner that two intersecting lines correspond to two triples having just one coordinate in common. The three sides of a triangle on the surface correspond to two such triples and their vector sum (mod 2).

To prove this theorem, we verify that the line $(0, b, c)$ is the line of intersection of the planes $(bc, 1, 0, 0)$ and $(0, 0, b, c)$. The three planes (abc, a, b, c), linearly dependent on these two, form with them the set of five planes which contain the given line $(0, b, c)$. In each plane there are two other lines:

	Plane	Lines
	$(bc, 1, 0, 0)$	$(0, \omega b, \bar{\omega} c), (0, \bar{\omega} b, \omega c)$
	$(0, 0, b, c)$	$(0, \omega b, \omega c), (0, \bar{\omega} b, \bar{\omega} c)$
(6)	$(bc, 1, b, c)$	$(1, 0, c), \quad (1, b, 0)$
	$(\omega bc, \omega, b, c)$	$(\omega, 0, c), \quad (\omega, b, 0)$
	$(\bar{\omega} bc, \bar{\omega}, b, c)$	$(\bar{\omega}, 0, c), \quad (\bar{\omega}, b, 0).$

These ten lines are characterized by the simple property that they each have just one coordinate in common with the line $(0, b, c)$. Furthermore, $(0, b, c)$ is the vector sum (mod 2) of the coordinates of each of the five pairs of these lines which intersect the line $(0, b, c)$ and form with it a triangle. Similar arguments for the lines $(a, 0, c)$ and $(a, b, 0)$ complete the proof of Theorem 2.

4. **The trihedral pairs.** If we now fix our attention on the forty non-isotropic planes of the $PG(3, 4)$, we find that each is orthogonal to twelve others. In this set of twelve planes, each is orthogonal to two others, which in turn are mutually orthogonal. Hence there are $40 \times 12 \times 2/4! = 40$ sets such as

$$(7) \qquad (1, 0, 0, 0), \quad (0, 1, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1),$$

consisting of four mutually orthogonal non-isotropic planes of the $PG(3, 4)$.

THEOREM 3. *The forty triads of trihedral pairs of tritangent planes to the general cubic surface are isomorphic with the forty sets of four mutually orthogonal non-isotropic planes of the* $PG(3, 4)$.

PROOF. Each such set of planes W, X, Y, Z in $PG(3, 4)$ may be divided in three ways into two pairs, such as W, X and Y, Z. The pair W, X determines three mutually non-orthogonal isotropic planes $W+kX$, $(k=1, \omega, \bar{\omega})$, which correspond to a triheder of three tritangent planes to the surface which do not intersect each other on the surface. The two associated pairs, W, X and Y, Z, correspond to a pair of conjugate triheders enjoying the property that each plane of the first cuts each plane of the second in a line of the surface. Since the two planes $W+kX$ and $W+lY$ are not mutually orthogonal, the three trihedral pairs formed from the set W, X, Y, Z have no common lines of the surface but exhaust the twenty-seven lines in three sets of nine lines.

5. **The group.** The group of unitary transformations of the $PG(3, 4)$, of determinant unity, transforms the forty triads of trihedral pairs among themselves. Furthermore, the four columns of the matrix of each transformation are a mutually orthogonal set of homogeneous coordinates for the transforms of the four non-isotropic planes (7). Altogether there are 4! permutations of the four columns, and 3^3 ways of multiplying the columns by factors without altering the value of the determinant. This proves that the order of the group is $40 \times 3^3 \times 4! = 25920$. A transformation of period two not contained in this group changes each coordinate to its conjugate mark in F. The direct product is a group of order 51840, which is isomorphic with the group of automorphisms of the twenty-seven lines on the general cubic surface.

BROWN UNIVERSITY

A SUFFICIENCY PROOF FOR ISOPERIMETRIC PROBLEMS IN THE CALCULUS OF VARIATIONS

M. R. HESTENES

The purpose of the present paper is to show that the sufficiency theorems for a strong relative minimum for isoperimetric problems can be obtained from those for simple integral problems with only a little additional argument. The method here used is a simple extension of one used by Birkhoff and Hestenes* for a special isoperimetric problem. Heretofore sufficiency theorems of this type have been obtained from those for a restricted relative minimum by the application of the theorem of Lindeberg. Sufficiency theorems for a restricted relative minimum can be obtained either from the theories of the problems of Lagrange and Bolza or by an argument analogous to that used for simple integrals.

The problem to be considered is that of minimizing an integral

$$J = \int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

in a class of admissible arcs

$$y_i(x), \qquad x_1 \leqq x \leqq x_2; \; i = 1, \cdots, n, \tag{1}$$

joining two fixed points 1 and 2 and satisfying a set of isoperimetric conditions

$$J_\alpha = \int_{x_1}^{x_2} f_\alpha(x, y, y')dx = l_\alpha, \qquad \alpha = 1, \cdots, m, \tag{2}$$

where the l's are constants. It will be assumed that the functions f, f_α are defined and have continuous derivatives of the first three orders in a region $\mathfrak{R}$ of points (x, y, y'). The points of $\mathfrak{R}$ will be called *admissible*. A continuous arc (1) that can be divided into a finite number of subarcs on each of which it has continuous derivatives will be called *admissible* if its elements (x, y, y') are all admissible.

Associated with the problem is an integral of the form

$$I_\lambda = \int_{x_1}^{x_2} F(x, y, y', \lambda)dx,$$

* *Natural isoperimetric conditions in the calculus of variations*, Duke Mathematical Journal, vol. 1 (1935), pp. 251–258. The method used in this paper was suggested by Professor Birkhoff.

where $F=f+\lambda_\alpha f_\alpha$* and the λ's are constant multipliers. It is in terms of this integral that the sufficiency conditions stated below are given. An admissible arc (1) and a set of constants λ_α having continuous second derivatives will be said to form an *isoperimetric extremal* if they satisfy the Euler-Lagrange equations

$$F_{y_i} - \frac{dF_{y_i'}}{dx} = 0. \tag{3}$$

Let E_0 be an isoperimetric extremal joining the points 1 and 2 and satisfying the conditions (2). It will be assumed that E_0 is normal, that is, that the equations $P_{i\alpha}a_\alpha = 0$, where

$$P_{i\alpha} = f_{\alpha y_i} - \frac{df_{\alpha y_i'}}{dx}, \tag{4}$$

hold along E_0 only in case the constants a_α are all zero. Subarcs of E_0 need not be normal. We shall suppose that the extremal E_0 has the following further properties. At each element (x, y, y', λ) in a neighborhood of those belonging to E_0 the inequality

$$F(x, y, Y', \lambda) - F(x, y, y', \lambda) - (Y_i' - y_i')F_{y_i'}(x, y, y', \lambda) > 0$$

holds for every admissible set $(x, y, Y') \neq (x, y, y')$. Moreover along E_0 the inequality $F_{y_i'y_k'}\pi_i\pi_k > 0$ holds for every set $(\pi) \neq (0)$. Finally

$$Q(\eta) = \int_{x_1}^{x_2} 2\omega(x, \eta, \eta')dx,$$

the second variation of I_λ along E_0, is positive for every non-null set of admissible variations $\eta_i(x)$, $(x_1 \leqq x \leqq x_2)$, vanishing at $x=x_1$ and $x=x_2$ and satisfying with E_0 the equations

$$L_\alpha(\eta) = \int_{x_1}^{x_2} \{f_{\alpha y_i}\eta_i + f_{\alpha y_i'}\eta_i'\}dx = 0. \tag{5}$$

An admissible variation $\eta_i(x)$, $(x_1 \leqq x \leqq x_2)$, is an arc having continuity properties like those of admissible arcs. The integrand 2ω appearing in $Q(\eta)$ is of the form $2\omega = F_{y_iy_k}\eta_i\eta_k + 2F_{y_iy_k'}\eta_i\eta_k' + F_{y_i'y_k'}\eta_i'\eta_k'$.

The theorem to be proved is the following:

THEOREM. *If E_0 has the properties described above, there is a neighborhood $\mathcal{N}$ of E_0 in xy-space such that the inequality $J(C) > J(E_0)$ holds for every admissible arc C in $\mathcal{N}$ joining* 1 *and* 2, *satisfying conditions* (2), *and not identical with E_0.*

* Repeated indices denote summation.

The proof of this result is based on two lemmas to be given below. In these lemmas we use the following definition of conjugate points. A value $x_3 \neq x_1$ is said to define a point 3 conjugate to 1 on E_0 relative to I_λ if there is a solution $\eta_i(x)$ of the equations

$$K_i(\eta) \equiv \omega_{\eta_i} - \frac{d\omega_{\eta_i'}}{dx} = 0 \tag{6}$$

having continuous second derivatives and vanishing at x_1 and x_3 but not identically zero on x_1x_3.

LEMMA 1. *If E_0 has on it no point 3 conjugate to 1 relative to I_λ, there exist neighborhoods $\mathcal{F}$ of E_0 in xy-space, N of the end values of E_0 in $(x_1y_1x_2y_2)$-space, and Λ of the multipliers λ_α belonging to E_0 such that for every set $(x_1y_1x_2y_2)$ in N and λ_α in Λ there is an isoperimetric extremal E_λ in $\mathcal{F}$ having $(x_1y_1x_2y_2)$ as its end values and λ_α as its multipliers. Moreover for these values of λ_α the inequality $I_\lambda(C) > I_\lambda(E_\lambda)$ holds for every admissible arc C in $\mathcal{F}$ joining the ends of E_λ and not identical with E_λ.*

This lemma is analogous to one given by Hahn and has been established by Birkhoff and Hestenes* following a method given by Bliss† in the proof of a similar theorem for the problem of Bolza. It should be noted that this lemma does not depend on the normality of E_0.

LEMMA 2. *Let $t_0 = x_1 < t_1 < \cdots < t_{q+1} = x_2$ be a set of values such that there are no pairs of conjugate points on E_0 relative to I_λ on any of the intervals $t_{r-1} \leqq x \leqq t_r$, $(r = 1, \cdots, q+1)$. There exists a qn parameter family of broken isoperimetric extremals*

$$y_i(x, b_{11}, \cdots, b_{nq}), \qquad \lambda_\alpha(b_{11}, \cdots, b_{nq}), \qquad x_1 \leqq x \leqq x_2, \tag{7}$$

containing E_0 for $b_{is} = b_{is0}$, satisfying the conditions (2), passing through the points 1, 2, and $(x, y_i) = (t_s, b_{is})$, $(s = 1, \cdots, q)$, and having no corners on the intervals $t_{r-1} < x < t_r$. The functions $y_i(x, b)$, $y_{ix}(x, b)$, $\lambda_\alpha(b)$ have continuous first and second derivatives for values (x, b) near those belonging to E_0 except possibly at the corner points. For the arc E_b of the family (7) determined by values $(b) \neq (b_0)$ in a sufficiently small neighborhood of $(b) = (b_0)$ one has $J(E_b) > J(E_0)$.

In order to establish this result let

$$Y_i(x, b_{11}, \cdots, b_{nq}, \lambda_1, \cdots, \lambda_m), \qquad \lambda_\alpha, \qquad x_1 \leqq x \leqq x_2, \tag{8}$$

* Loc. cit., pp. 253–254.

† *The problem of Bolza in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 267–270.

be a $(qn+m)$ parameter family of broken extremals containing E_0 for values $b_{is}=b_{is0}$, $\lambda_\alpha=\lambda_{\alpha 0}$, passing through the points 1, 2, and $(x, y_i)=(t_s, b_{is})$ and having no corners on the intervals $t_{r-1}<x<t_r$. Except at the corner points the functions $Y_i(x, b, \lambda)$, $Y_{is}(x, b, \lambda)$ have continuous first and second derivatives for values (x, b, λ) near those belonging to E_0. The existence of a family of this type follows readily from the first part of Lemma 1 and existence theorems for differential equations of the form (3). When the functions $Y_i(x, b, \lambda)$ are substituted in the integrals (2) a set of functions $J_\alpha(b, \lambda)$ is obtained having continuous first and second derivatives for values (b, λ) near those on E_0. The functional determinant $|\partial J_\alpha/\partial\lambda_\beta|$, $(\alpha, \beta=1, \cdots, m)$, is different from zero when $(b, \lambda)=(b_0, \lambda_0)$, as will be seen in the next paragraph. The equations $J_\alpha(b, \lambda)=l_\alpha$ are satisfied by the values $(b, \lambda)=(b_0, \lambda_0)$ and hence have unique solutions $\lambda_\alpha(b)$, with $\lambda_\alpha(b_0)=\lambda_{\alpha 0}$, having continuous second derivatives near $(b)=(b_0)$. When the functions $\lambda_\alpha(b)$ are substituted for λ_α in the functions (8), a family (7) is obtained having the properties described in the lemma. The first part of the lemma is immediate. In order to prove the last part we note that by virtue of the identities

$$y_i(x_1, b) = y_{i1}, \qquad y_i(t_s, b) = b_{is}, \qquad y_i(x_2, b) = y_{i2},$$

where (x_1, y_1), (x_2, y_2) are the points 1 and 2, respectively, the variations

$$\delta y_i(x) = y_{ib_{js}}(x, b_0)db_{js}, \qquad j = 1, \cdots, n; s = 1, \cdots, q,$$

satisfy the relations

$$\delta y_i(x_1) = 0, \qquad \delta y_i(t_s) = db_{is}, \qquad \delta y_i(x_2) = 0,$$

so that $\delta y_i(x)\not\equiv 0$ on x_1x_2 if $(db)\neq(0)$. Moreover the variations $\eta_i=\delta y_i$ satisfy equations (5). For by construction the value $J_\alpha(b)$ of the integral J_α along the family (7) is a constant. It follows that $dJ_\alpha=L_\alpha(\delta y)=0$ along E_0. From the assumptions on E_0 we have accordingly $Q(\delta y)>0$ whenever $(db)\neq(0)$. Consider now the function

$$J_0(b) = J(b) + \lambda_{\alpha 0}J_\alpha(b)$$

obtained by evaluating the integral $J_0=J+\lambda_{\alpha 0}J_\alpha$ along the arc $y_i(x, b)$ of the family (7). By the use of the Euler-Lagrange equations (3) and the properties of δy_i just established, it is found that along E_0 one has

$$dJ_0 = \int_{x_1}^{x_2} \{F_{y_i}\delta y_i + F_{y_i'}\delta y_i'\}\,dx = F_{y_i'}\delta y_i\Big|_{x_1}^{x_2} = 0, \qquad d^2J_0 = Q(\delta y) > 0$$

for all values $(db) \neq (0)$. In view of these relations and the equations $J_\alpha(b) = l_\alpha$ it follows that for values $(b) \neq (b_0)$ near $(b) = (b_0)$ the inequality

$$0 < J_0(b) - J_0(b_0) = J(b) - J(b_0)$$

holds, as was to be proved.

The proof of Lemma 2 will be complete if we show that the determinant $|\partial J_\alpha/\partial \lambda_\beta|$ is different from zero at $(b, \lambda) = (b_0, \lambda_0)$. To prove this let $\delta' Y_i(x) = Y_{i\lambda_\beta}(x, b_0, \lambda_0) d\lambda_\beta$. For the function $J_\alpha(b, \lambda)$ one then has along E_0

$$\delta' J_\alpha = \frac{\partial J_\alpha}{\partial \lambda_\beta} d\lambda_\beta = L_\alpha(\delta' Y), \tag{9}$$

where the functions L_α are given by equations (5). From the relations

$$Y_i(x_1, b, \lambda) = y_{i1}, \qquad Y_i(t_s, b, \lambda) = b_{is}, \qquad Y_i(x_2, b, \lambda) = y_{i2}$$

it follows by differentiation that

$$\delta' Y_i(x_1) = 0, \qquad \delta' Y_i(t_s) = 0, \qquad \delta' Y_i(x_2) = 0. \tag{10}$$

Moreover along E_0 we have

$$K_i(\delta' Y) + P_{i\beta} d\lambda_\beta = 0, \tag{11}$$

where $P_{i\beta}$, K_i are given by equations (4) and (6). This can be seen by substituting the functions (8) in the Euler-Lagrange equations (3), differentiating for λ_β, multiplying by $d\lambda_\beta$, and summing. Suppose now that the determinant in question were zero. Then there would exist constants $d\lambda_\beta$ not all zero such that $\delta' J_\alpha = 0$ along E_0. If $\delta' Y_i \equiv 0$ on $x_1 x_2$, one would have $P_{i\beta} d\lambda_\beta \equiv 0$ on $x_1 x_2$ by equations (11), and E_0 could not be normal. Hence $\delta' Y_i \not\equiv 0$ on $x_1 x_2$. By the use of equations (9), (10), (11), and $\delta' J_\alpha = 0$ and integration by parts it would be found that along E_0 we would have $L_\alpha(\delta' Y) = 0$ and

$$0 = \int_{x_1}^{x_2} \delta' Y_i [K_i(\delta' Y) + P_{i\beta} d\lambda_\beta] dx = Q(\delta' Y) + \delta' J_\beta d\lambda_\beta = Q(\delta' Y),$$

contrary to our assumption concerning the value of the second variation $Q(\eta)$ along E_0. Hence $|\partial J_\alpha/\partial \lambda_\alpha| \neq 0$ on E_0 and Lemma 2 is established.

We are now in position to prove Theorem 1. To do so we let $\mathcal{F}'$ be a neighborhood of E_0 in xy-space so small that each subarc of the family (7) in $\mathcal{F}'$ with end points on successive hyperplanes $x = t_{r-1}$, $x = t_r$ affords a minimum to the integral I_λ relative to admissible arcs in $\mathcal{F}'$

joining its end points. This is possible by virtue of Lemma 1. Let $\mathcal{Y}$ be a second neighborhood of E_0 interior to $\mathcal{Y}'$ such that every admissible arc C in $\mathcal{Y}$ joining the points 1 and 2 and satisfying equations (2) cuts the hyperplanes $x=t_i$ in points (t_i, b_{i_0}) whose y-coordinates b_{i_0} determine an extremal E_b of the family (7) lying in $\mathcal{Y}'$. By Lemma 1 we have $I_\lambda(C) \geqq I_\lambda(E_b)$, the multipliers λ_α being those belonging to E_b. But since the arcs C and E_b satisfy equations (2), this implies that

$$I_\lambda(C) - I_\lambda(E) = J(C) - J(E_b) \geqq 0,$$

the equality holding only in case $C \equiv E_b$. Diminish $\mathcal{Y}$ if necessary so that $J(E_b) \geqq J(E_0)$, as described in Lemma 2. We then have $J(C) \geqq J(E_b) \geqq J(E_0)$, the equality holding in both cases only in case $C \equiv E_0$. This proves the theorem.

THE UNIVERSITY OF CHICAGO

A NEW SUMMATION METHOD FOR DIVERGENT SERIES*

W. A. MERSMAN

1. **Introduction.** The method to be given here is a modification of that due to Euler-Knopp.† For the weighted means of the partial sums we use the binomial coefficients, but instead of beginning with the first we begin with the "central" one, that is with the greatest. Thus the initial terms always receive the greatest weight, as in the Cesàro-Hölder method.

In this paper it is shown (1) that this new method includes the first two Cesàro methods, and (2) that it also includes the first Euler-Knopp method; further, (3) the exact range of summability of the geometric series is determined. Finally, an example is given which indicates that this method may be more powerful than all those of Cesàro-Hölder, although this statement has not yet been proved.

2. **Definitions and notation.** Throughout we consider a series $\sum_{k=0}^{\infty} a_k$ and denote by S_n the sum of its first $n+1$ terms. We define σ_n as follows:

$$\sigma_n = \frac{1}{4^n} \sum_{k=0}^{n} C_{2n+1,n-k} S_k, \tag{1}$$

where $C_{n,k}$ denotes the ordinary binomial coefficient. If σ_n approaches

* Presented to the Society, April 11, 1936. See abstract 42-5-139.

† K. Knopp, Mathematische Zeitschrift, vol. 15 (1922), pp. 226–253.

a limit as $n \to \infty$, the original series is said to be summable by our new method.

The regularity of this new method need not be proved directly since it follows from Theorem 1 or Theorem 2 below.

3. **Relation to Cesàro's method.** In this and the following section we shall make use of a theorem due to Silverman* and Toeplitz:†

If two sequences t_n and s_n are related by the equations

$$t_n = \sum_{k=0}^{n} a_{nk} s_k,$$

then necessary and sufficient conditions that $t_n \to s$ whenever $s_n \to s$ are that

(a) $\lim_{n\to\infty} a_{nk} = 0$, *for any* k,
(b) $\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk} = 1$,
(c) $\sum_{k=0}^{n} |a_{nk}| < M$, *independent of* n.

We begin by recalling the definition of the Cesàro method: If

$$A_n^r = \sum_{m=0}^{n} C_{r+m-1,m} S_{n-m},$$

and if

$$\alpha_n^r = A_n^r / C_{r+n,n},$$

the series is said to be summable (C, r) if α_n^r approaches a limit as $n \to \infty$, r remaining fixed.

Equating coefficients in the identity‡

$$(1 - x)^r \sum_{n=0}^{\infty} A_n^r x^n = \sum_{n=0}^{\infty} S_n x^n, \tag{2}$$

we obtain

$$S_n = \sum_{k=0}^{r} (-1)^k C_{r,k} C_{r+n-k,r} \alpha_{n-k}^r,$$

and substitution in the definition (1) gives the expression for our means as linear functions of Cesàro's, which enables us to apply directly the theorem of Silverman and Toeplitz. We obtain

* Doctor's thesis, University of Missouri Studies, Mathematics, ser. 1 (1913), pp. 1–96.

† Prace Matematyczno-Fizyczne, vol. 22 (1911), pp. 113–119.

‡ S. Chapman, Proceedings of the London Mathematical Society, vol. 9 (1910–1911), pp. 369–409.

$$\sigma_m = \sum_{j=0}^{m} a_{mj}^{r} \alpha_j^{r}, \tag{3}$$

where

$$a_{mj}^{r} = \frac{1}{4^m} \sum_{n=j}^{j+r} (-1)^{n-j} C_{2m+1,m-n}\, C_{r,n-j}\, C_{r+j,j}. \tag{4}$$

We have now to prove that the matrix $\|a_{mj}^{r}\|$ satisfies conditions (a), (b), (c), for $r=2$. We shall verify (a) and (b) for any positive r, and (c) for $r=2$.

PROOF OF (a). We need the asymptotic relation

$$\frac{1}{2^{2m+1}} C_{2m+1,m} \to (m\pi)^{-1/2}, \tag{5}$$

which can be proved either by means of Stirling's theorem* or by means of a result due to Knopp.† Hence, from (4),

$$\begin{aligned} |a_{mj}^{r}| &\leqq \frac{1}{4^m} C_{r+j,j} \sum_{n=j}^{j+r} C_{2m+1,m-n}\, C_{r,n-j} \\ &\leqq \frac{1}{4^m} C_{r+j,j}\, C_{2m+1,m} \sum_{n=j}^{j+r} C_{r,n-j} \\ &= \frac{1}{4^m} C_{r+j,j}\, C_{2m+1,m} \cdot 2^r \to 0. \end{aligned}$$

PROOF OF (b). Again from (4),

$$\sum_{j=0}^{m} a_{mj}^{r} = \frac{1}{4^m} \sum_{j=0}^{m} \sum_{n=j}^{j+r} (-1)^{n-j} C_{2m+1,m-n}\, C_{r,n-j}\, C_{r+j,j}.$$

Interchanging the order of summation, we obtain the relation

$$\sum_{j=0}^{m} a_{mj}^{r} = \frac{1}{4^m} \sum_{n=0}^{m} (-1)^n C_{2m+1,m-n} \sum_{j=0}^{n} (-1)^j C_{r+j,j} C_{r,n-j}.$$

Now

$$\sum_{j=0}^{n} (-1)^j C_{r+j,j} C_{r,n-j} = (-1)^n,$$

as is easily seen by equating coefficients in the identity

* E. C. Titchmarsh, *The Theory of Functions*, §1.87.

† K. Knopp, *Theory and Application of Infinite Series*, p. 384, example 3.

$$(1-x)^r\cdot(1-x)^{-r-1}=\sum_{n=0}^{\infty}x^n.$$

Thus

$$\sum_{j=0}^{m}a_{mj}^{r}=\frac{1}{4^m}\sum_{n=0}^{m}C_{2m+1,m-n}=1, \tag{6}$$

since the sum is exactly half the sum $\sum_{n=0}^{2m+1}C_{2m+1,n}=2^{2m+1}$.

PROOF OF (c) WITH $r=2$. After putting $r=2$ in (4) and writing out all the terms in the sum, we find that

$$a_{mj}^{2}=\frac{(2m+1)!(j+1)(j+2)(-m+2j^2+6j+3)}{[4^m(m-j)!(m+j+3)!]}, \tag{7}$$

so that a_{mj}^{2} is negative or positive according as

$$j\leqq k, \qquad j\geqq k, \qquad\qquad k=\left(\frac{m}{2}+\frac{3}{4}\right)^{1/2}-\frac{3}{2}.$$

Hence

$$\sum_{j=0}^{m}|a_{mj}^{2}|=1+\frac{4(2m+1)!}{2^{2m+1}}\sum_{j=0}^{k}\frac{(j+1)(j+2)(m-2j^2-6j-3)}{(m-j)!(m+j+3)!}$$

$$\leqq 1+\frac{4}{2^{2m+1}}C_{2m+1,m}\sum_{j=0}^{k}\frac{(j+1)(j+2)(m-2j^2-6j-3)}{(m+2)(m+3)}.$$

Now*

$$\sum_{j=0}^{k}j^4=O(k^5)=O(m^{5/2}).$$

Therefore

$$\sum_{j=0}^{m}|a_{mj}^{2}|\leqq 1+\frac{4}{2^{2m+1}}C_{2m+1,m}\cdot O(m^{1/2}),$$

and, by (5), the right-hand member is bounded. This completes our proof.

Thus we have established the following result:

THEOREM 1. *Any series summable* $(C, 2)$ *is summable by this new method to the same sum.*

4. **Relation to the method of Euler-Knopp.** The definition of the Euler-Knopp method is as follows. If

* Cf. E. Lucas, *Théorie des Nombres*, p. 238, §134, formula 4.

$$E_n = \frac{1}{2^n}\sum_{k=0}^{n} C_{n,k}S_k,$$

the series is said to be summable $(E, 1)$ if E_n approaches a limit as $n \to \infty$.

As Rey-Pastor* has shown,

$$S_n = \sum_{r=0}^{n} (-1)^{n-r}C_{n,r}2^r E_r.$$

Substitution in (1) gives our means as linear functions of those of Euler-Knopp:

$$\sigma_m = \sum_{r=0}^{m} a_{mr}E_r, \tag{8}$$

where

$$a_{mr} = \frac{1}{4^m}\sum_{n=r}^{m} (-1)^{n-r}C_{n,r}C_{2m+1,m-n}2^r. \tag{9}$$

Applying the recursion relation

$$C_{n,k} = C_{n-1,k-1} + C_{n-1,k},$$

and proceeding by induction, we obtain

$$a_{mr} = \frac{2^r}{4^m}C_{2m-r,m} > 0. \tag{10}$$

PROOF OF (a). Use (5) and (10).

PROOF OF (b) AND (c). Because of (10), (c) follows from (b). Now, from (9),

$$\sum_{r=0}^{m} a_{mr} = \frac{1}{4^m}\sum_{r=0}^{m}\sum_{n=r}^{m} (-1)^{n-r}C_{n,r}C_{2m+1,m-n}2^r.$$

Interchanging the order of summation, we have, by (6),

$$\sum_{r=0}^{m} a_{mr} = \frac{1}{4^m}\sum_{n=0}^{m} C_{2m+1,m-n}\sum_{r=0}^{n} (-1)^{n-r}C_{n,r}2^r$$

$$= \frac{1}{4^m}\sum_{n=0}^{m} C_{2m+1,m-n}(2-1)^n = 1.$$

* Publicaciones de la Facultad de las Ciencias Exactas, Físicas, y Naturales de la Universidad de Buenos Aires, (B), no. 12 (1932), pp. 51–222.

The result we have just established may be expressed as follows:

THEOREM 2. *Any series summable* $(E, 1)$ *is summable by this new method to the same sum.*

It may be remarked that, since neither $(C, 2)$ nor $(E, 1)$ summation includes the other, this new method is more general than either of them.

5. **The geometric series and other examples.** Consider the geometric series, with complex variable z,

$$\sum_{k=0}^{\infty} z^k, \qquad S_m = \sum_{k=0}^{m} z^k = \frac{1 - z^{m+1}}{1 - z}. \tag{11}$$

Then from (1),

$$\sigma_n(z) = \frac{1}{4^n} \sum_{m=0}^{n} C_{2n+1,n-m} \frac{1 - z^{m+1}}{1 - z},$$

or

$$\sigma_n(z) = S(z) + R_n(z), \tag{12}$$

where

$$S(z) = \frac{1}{1 - z}, \qquad R_n(z) = \frac{z}{z - 1} \cdot \frac{1}{4^n} \sum_{m=0}^{n} C_{2n+1,n-m} z^m. \tag{13}$$

From the binomial expansion of $[(z+1)/2z^{1/2}]^{2n+1}$ we find that

$$R_n(z) = \frac{2z^{1/2}}{z - 1} \left(\frac{z + 1}{2z^{1/2}} \right)^{2n+1} + R_n\left(\frac{1}{z} \right). \tag{14}$$

Since the geometric series (11) is summable $(C, 1)$ to the sum $S(z) = 1/(1-z)$ and hence, by Theorem 1, summable by our method to the same sum, it follows that $R_n(1/z) \to 0$, $(|z| \geq 1,\ z \neq 1)$. Therefore, from (12), (13), (14) we have the following theorem:

THEOREM 3. *The geometric series* $\sum_{k=0}^{\infty} z^k$, z *complex, is summable by this new method to the sum* $1/(1-z)$ *in, and only in, the interior of the closed curve whose equation is*

$$\left| \frac{z + 1}{2z^{1/2}} \right| = 1, \qquad \text{or} \qquad r^2 + 2r \cos \theta - 4r + 1 = 0. \tag{15}$$

In particular, for $z = -x$, x real and positive, the range of summability is $-1 < x < 3 + 2^{3/2}$.

Also, it is easily seen that the point $z=3/2+(-7)^{1/2}/2$ lies on the curve (15); that is, the region of summability extends to the right of the line $R(z)=1$. This shows that no Euler-Knopp method* of any order can include this new method. The same is true of the Cesàro methods, since the region of (C, r) summability of the geometric series is $|z| \leqq 1, z \neq 1$, no matter how large r may be.

We conclude with the example mentioned in the introduction. Consider the sequence

$$S_n = (-1)^n C_{n+r,r}, \tag{16}$$

r fixed. From the identity (2) and the Maclaurin expansion of $(1-x)^{-r-1}$ we find (in the notation of §3) that

$$A_{2n}^r = -A_{2n+1}^r = C_{n+r,r}. \tag{17}$$

Hence α_n^r oscillates between $1/2^r$ and $-1/2^r$, so that the sequence (16) is not limitable (C, r). But from (17),

$$A_{2n}^{r+1} = C_{n+r,r}, \qquad A_{2n+1}^{r+1} = 0, \tag{18}$$

so that the sequence (16) is limitable $(C, r+1)$ to zero.

We now show that, for any fixed r, (16) is limitable by the new method. In fact, from (1),

$$\sigma_n = \frac{1}{4^n} \sum_{m=0}^{n} C_{2n+1,n-m}(-1)^m C_{m+r,r}.$$

A proof similar to that of (10) above shows that

$$\sigma_n = \frac{1}{4^n} C_{2n-r,n},$$

and (5) above completes the proof.

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* K. Knopp, *Theory and Application of Infinite Series*, p. 510.

METRIC PROPERTIES OF THE CYLINDER OF KUBOTA

RUTH B. RASMUSEN

1. **Introduction.** It is the purpose of this note to derive some properties of the surface normal and the *superosculating lines* on an analytic surface S by means of a parabolic cylinder which Kubota has defined* and has used for studying the properties of the affine normal to a surface and the curves of Darboux and Segre in affine differential geometry. If we consider all of the sections of an analytic surface S made by planes passing through the same tangent t, the locus of the parabolas which osculate these sections at the common point of contact P is a parabolic cylinder which we shall call the *cylinder of Kubota*.

In what follows we shall adhere quite closely to the notation used in Chapter 6 of Lane's *Projective Differential Geometry of Curves and Surfaces*, Chicago, 1932.

2. **Analytic basis.** It is convenient to take the lines of curvature for the parametric curves and to employ a local trihedron at a point of the surface whose edges are the tangents of the lines of curvature and the normal of the surface at the point. This section is designed to introduce these concepts and to collect some formulas which will be used later on in this note.

Let us consider in ordinary metric space a non-developable surface, not a sphere, whose parametric equations in cartesian coordinates are

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

Let the lines of curvature be the parametric curves on this surface. Then its first and second fundamental forms, written in the customary notation, are

$$E\,du^2 + G\,dv^2, \qquad D\,du^2 + D''dv^2.$$

The principal radii of normal curvature R_1, R_2 at a point of the surface are defined by the formulas $R_1 = E/D$, $R_2 = G/D''$.

As a local trihedron of reference at a point (x, y, z) of the surface S,

* T. Kubota, *Einige Bemerkungen zur Affinflächentheorie*, The Science Reports of the Tôhoku Imperial University, (1), vol. 19 (1930), p. 163. See also T. Kubota, *Einige Bemerkungen zur Affinflächentheorie*, Japanese Journal of Mathematics, vol. 10 (1933), p. 217.

we shall take the origin at this point, the ξ-axis along the u-tangent, the η-axis along the v-tangent, and the ζ-axis along the surface normal.

The power series expansions for the local coordinates ξ, η, ζ of a point near a point P (0, 0, 0) on a surface S in terms of the increments Δu, Δv corresponding to displacement on S from the point (0, 0, 0) to the point (ξ, η, ζ) are

$$
\begin{aligned}
\xi &= E^{1/2}[\Delta u + \tfrac{1}{2}(\{11, 1\}\Delta u^2 + 2\{12, 1\}\Delta u\,\Delta v + \{22, 1\}\Delta v^2) + \cdots], \\
\eta &= G^{1/2}[\Delta v + \tfrac{1}{2}(\{11, 2\}\Delta u^2 + 2\{12, 2\}\Delta u\,\Delta v + \{22, 2\}\Delta v^2) + \cdots], \\
\zeta &= \tfrac{1}{2}(D\Delta u^2 + D''\Delta v^2) + \tfrac{1}{6}[(D_u + \{11, 1\}D)\Delta u^3 + 3\{12, 1\}D\Delta u^2\Delta v \\
&\quad + 3\{12, 2\}D''\Delta u\Delta v^2 + (D_v'' + \{22, 2\}D'')\Delta v^3] + \cdots .
\end{aligned} \tag{1}
$$

From the expansions (1) it is possible to compute an expansion for ζ as a power series in ξ, η with undetermined coefficients and then demand that the expansions for ξ, η, ζ in (1) shall satisfy this equation identically in Δu, Δv as far as the terms of any desired order. Thus we find, to terms of the third order,

$$
\zeta = \frac{1}{2}\left(\frac{\xi^2}{R_1} + \frac{\eta^2}{R_2}\right) + \frac{1}{6}(\alpha_0\xi^3 + 3\alpha_1\xi^2\eta + 3\alpha_2\xi\eta^2 + \alpha_3\eta^3) + \cdots , \tag{2}
$$

where

$$
\alpha_0 = \frac{1}{E^{1/2}}\left(\frac{1}{R_1}\right)_u, \qquad \alpha_1 = \frac{1}{G^{1/2}}\left(\frac{1}{R_1}\right)_v, \qquad \alpha_2 = \frac{1}{E^{1/2}}\left(\frac{1}{R_2}\right)_u,
$$

$$
\alpha_3 = \frac{1}{G^{1/2}}\left(\frac{1}{R_2}\right)_v .
$$

3. **The equation of the cylinder of Kubota.** In order to find the equation of the parabolic cylinder of Kubota, let us regard a curve C as imbedded in the one-parameter family of curves on the surface S defined by the equation $dv - \lambda du = 0$, where λ is a function of u, v. The tangent t of the curve C at point P has the equations $\zeta = \eta - A\xi = 0$, where

$$
A = \lambda\left(\frac{G}{E}\right)^{1/2} .
$$

If we cut the surface (2) by the plane

$$
\zeta = n(\eta - A\xi), \tag{3}
$$

not passing through an asymptotic tangent, then the equation of the parallel projection of the curve of section in the direction of the ζ-axis on the tangent plane is

$$n(\eta - A\xi) = \frac{1}{2}\left(\frac{\xi^2}{R_1} + \frac{\eta^2}{R_2}\right) + \frac{1}{6}\phi_3(\xi, \eta) + \cdots . \tag{4}$$

It is convenient to make the following definition of $\phi_3(\xi, \eta)$:

$$\phi_3(\xi, \eta) = \alpha_0\xi^3 + 3\alpha_1\xi^2\eta + 3\alpha_2\xi\eta^2 + \alpha_3\eta^3.$$

If $n \neq 0$, equation (4) is representable as a power series in the form

$$\eta = A\xi + B\xi^2 + C\xi^3 + \cdots , \tag{5}$$

in which the coefficients have the following values:

$$A = \lambda\left(\frac{G}{E}\right)^{1/2}, \quad nB = \frac{1}{2}\left(\frac{1}{R_1} + \frac{A^2}{R_2}\right), \quad nC = \frac{AB}{R_2} + \frac{1}{6}\phi_3(1, A).$$

The equation of the osculating parabola at P of the curve (5) is

$$p\xi^2 + q\xi\eta + r\eta^2 + \eta - A\xi = 0, \tag{6}$$

in which the coefficients are given by

$$p = A^2r + (AC - B^2)/B, \quad q = -(2ABr + C)/B, \quad r = -C^2/4B^3.$$

If we eliminate n between equations (3) and (6), we obtain

$$[3(\xi/R_1 + A\eta/R_2)(1/R_1 + A^2/R_2) + \phi_3(1, A)\zeta]^2 - 18\zeta(1/R_1 + A^2/R_2)^3 = 0, \tag{7}$$

which is the *equation of the parabolic cylinder of Kubota.*

4. **Properties.** The equation of the diametral plane of cylinder (7) which passes through the point P is

$$3(\xi/R_1 + A\eta/R_2)(1/R_1 + A^2/R_2) + \phi_3(1, A)\zeta = 0. \tag{8}$$

We now proceed to prove the following theorem:

If the tangent t is permitted to vary about the point P, the diametral plane passing through the point P of this parabolic cylinder envelops an algebraic cone of the fourth order and the third class with three cusp edges which intersect in the line of centers of the quadrics of Darboux. Three tangent planes intersect in the surface normal. The three directions of the tangent t to which these tangent planes correspond are the tangents of the superosculating lines, and these tangent planes give as the lines of intersection with the tangent plane the harmonic conjugates of the tangents of the superosculating lines.

In order to obtain the equation of the *enveloping cone of plane* (8) *in tangential coordinates*, we set

$$\rho u_1 = (1/R_1 + A^2/R_2)/R_1,$$
$$\rho u_2 = A(1/R_1 + A^2/R_2)/R_2,$$
$$\rho u_3 = \phi_3(1, A)/3.$$

Eliminating ρ and λ homogeneously from the above equations, we obtain

$$3G^{1/2}R_1(GR_1u_1^2 + ER_2u_2^2)u_3 = \phi_3(G^{1/2}R_1u_1, E^{1/2}R_2u_2).$$

Consequently the cone is of class three.

The equation of the enveloping cone of plane (8) in point coordinates is obtained by eliminating λ between equation (8) and the equation obtained by differentiating equation (8) with respect to λ. The *equation of the cone in point coordinates* is

$$(9) \qquad \left(a\frac{E}{G}\right)^3 - [4b^3d - 27a^2d^2 + b^2c^2 - 4ac^3 + 18abcd] = 0,$$

in which a, b, c, d are defined by

$$a = \frac{\eta}{R_2^2} + \frac{\alpha_3\zeta}{3}, \qquad b = \frac{\xi}{R_1R_2} + \alpha_2\zeta, \qquad c = \frac{\eta}{R_1R_2} + \alpha_1\zeta,$$
$$d = \frac{\xi}{R_1^2} + \frac{\alpha_0\zeta}{3}.$$

Neglecting the factor a, we see that the cone is of the fourth degree in ξ, η, ζ. The planes $a=0$ and $d=0$ are tangent planes, and the lines $a=b=0$ and $c=d=0$ are generators, of cone (9).

When a surface is referred to its lines of curvature, the curves which possess the property that the normal sections in the directions of these curves at a point are straight lines, or are superosculated by their circles of curvature, are called the *superosculating lines** of the surface. By direct computation it can be shown that as $n\to\infty$ the directions for which curve (4) is superosculated by its osculating circle are given by

$$\phi_3(1, A) = 0.$$

Using the definition of the superosculating lines, and verifying that the harmonic conjugate of $\zeta=\eta-A\xi=0$ is $\zeta=\xi/R_1+A\eta/R_2=0$, we obtain the above theorem.

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* Eisenhart, *Differential Geometry*, Boston, 1909, p. 187.

APOLARITY OF TRILINEAR FORMS AND PENCILS OF BILINEAR FORMS*

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1. **Introduction.** In this note the idea of apolar trilinear forms, introduced earlier by the author,† is generalized, and the main theorems are given new proofs independent of the group theoretic methods and notation used then. The importance of the apolarity concept in trilinear form classification is illustrated by a numerical application. In §3 there is given a new method of classifying singular pencils of bilinear forms based on Dickson's minimal numbers.‡

2. **Apolarity of trilinear forms.** A trilinear form $F(x, y, z) = \sum a_{hij}x_h y_i z_j$, where h runs from 1 to l, i from 1 to m, and j from 1 to n, has as *two-way rank invariants*§ r_h, r_i, r_j the smallest numbers of variables x, y, z, respectively, in terms of which the form can be expressed. Let F and $F' = \sum b_{hij}x_h y_i z_j$ be two trilinear forms in which the numbers of x's, y's, z's are r_h, m, n and r_h', m, n, respectively, where r_h and r_h' are the two-way h-rank invariants of F and F'.

DEFINITION. *F and F' are said to be h apolar (relative to m, n)‖ if*

$$\sum_{i,j=1}^{m,n} a_{hij}b_{\nu ij} = 0, \qquad h = 1, \cdots, r_h;\ \nu = 1, \cdots, r_h', \tag{1}$$

and

$$r_h + r_h' = mn. \tag{2}$$

We define i and j apolarity analogously with respect to the i and j two-way rank invariants of F and F'. We shall apply the term *apolar* to two forms if they are h, i, or j apolar. The following theorems concerning h apolarity can, of course, be rephrased in terms of i and j apolarity.

We consider questions of existence and uniqueness properties of

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* Presented to the Society, April 8, 1938.

† *Metabelian groups and trilinear forms*, American Journal of Mathematics, vol. 60 (1938), pp. 383–415.

‡ *Singular case of pairs of bilinear, quadratic, or Hermitian forms*, Transactions of this Society, vol. 29 (1927), pp. 239–253.

§ Cf. R. Oldenburger, *On canonical binary trilinear forms*, this Bulletin, vol. 38 (1932), p. 385.

‖ Evidently $m \geqq r_i$, $n \geqq r_j$. For most applications we take $m = r_i$, $n = r_j$.

forms F' apolar to a given form F (relative to m, n). Let $a_{hij}=a_{h\lambda}$, where $\lambda=(i-1)n+j$.* Defining $b_{\nu ij}=b_{\nu\lambda}$, we can write (1) as

$$(1') \qquad \sum_{\lambda=1}^{mn} a_{h\lambda}b_{\nu\lambda} = 0, \qquad h = 1, \cdots, r_h; \nu = 1, \cdots, r_h = mn - r_h.$$

The rank of $(a_{h\lambda})$ is r_h, for if it were less, F could be expressed in terms of less than r_h variables x. The $b_{\nu\lambda}$ in A' can be interpreted as a set of r_h' linearly independent solutions of the equations $\sum_{\lambda=1}^{mn} a_{h\lambda}l_\lambda=0$, $(h=1, \cdots, r_h)$. It is well known that such solutions $b_{\nu\lambda}$ always exist, and that furthermore, if $c_{\mu\lambda}$ is a second set of solutions, constants $k_{\mu\nu}$ exist such that

$$(3') \qquad c_{\mu\lambda} = \sum_{\nu=1}^{r_h'} k_{\mu\nu}b_{\nu\lambda}, \qquad \mu = 1, \cdots, r_h',$$

where $|k_{\mu\nu}| \neq 0$.† Formula (1) is the usual definition of apolarity of two-way matrices. If we write $c_{\mu ij}=c_{\mu\lambda}$, (3′) becomes

$$(3) \qquad c_{\mu ij} = \sum_{\nu=1}^{r_h'} k_{\mu\nu}b_{\nu ij}, \qquad \mu = 1, \cdots, r_h'.$$

If we subject the x_ν in F' to the non-singular linear transformation $x_\nu=\sum_\mu k_{\mu\nu}x_\mu'$, F' becomes

$$\sum_{\nu,i,j} b_{\nu ij}\left(\sum_\mu k_{\mu\nu}x_\mu'\right)y_i z_j = \sum_{\mu,i,j}\left(\sum_\nu k_{\mu\nu}b_{\nu ij}\right)x_\mu' y_i z_j = \sum_{\mu,i,j} c_{\mu ij}x_\mu' y_i z_j.$$

We have proved the following theorem:

THEOREM 1. *For every trilinear form F with invariants r_h, r_i, r_j and every pair of integers m $(\geqq r_i)$, n $(\geqq r_j)$ there exists a trilinear form F' which is h apolar to F. Any second form F'' which is h apolar to F (relative to m, n) is conjugate to F' under non-singular linear transformations on the variables x_ν of F'.*

Since the steps are all rational, the coefficients of F' can be chosen from the field of the coefficients of F.

Two trilinear forms F and G are said to be *equivalent*, written $F\sim G$, if F can be sent into G by non-singular linear transformations on its sets of variables x, y, z, taken separately.

* The matrix $(a_{h\lambda})$ is called the *two-way h-display* of F. See R. Oldenburger, *Non-singular multilinear forms and certain p-way matrix factorizations*, Transactions of this Society, vol. 39 (1936), pp. 423–424.

† See, for example, J. H. M. Wedderburn, *Lectures on Matrices*, American Mathematical Society Colloquium Publications, vol. 17, New York, 1934, p. 19, Theorem 10.

THEOREM 2. *If $F \sim G$, and if F', G' are h apolar to F, G, respectively, (relative to the same numbers m, n), then $F' \sim G'$.*

If F can be sent into G by transformations on x alone, then F is, by Theorem 1, h apolar to G'; whence $F' \sim G'$. Next suppose that $G = \sum c_{h\mu j} x_h y'_\mu z_j$ is obtained from F by the transformation T, $y_i = \sum k_{\mu i} y'_\mu$; that is, $c_{h\mu j} = \sum_i k_{\mu i} a_{hij}$. Then let $F^* = \sum d_{\nu\rho j} x_\nu y'_\rho z_j$ be obtained from $F' = \sum b_{\nu\rho j} x_\nu y_\rho z_j$ by applying the transformation $y_\rho = \sum \kappa_{\mu\rho} y'_\mu$, contragredient to T, that is, $\sum \kappa_{\mu\rho} k_{\mu i} = \delta_{\rho i}$ (Kronecker's δ_{ij}). We see that F^* is h apolar to G, for

$$\sum_{\mu,j} d_{\nu\mu j} c_{h\mu j} = \sum_{i,\rho,j} \left(\sum_{\mu} \kappa_{\mu\rho} k_{\mu i} \right) b_{\nu\rho j} a_{hij} = \sum_{i,\rho,j} \delta_{i\rho} b_{\nu\rho j} a_{hij},$$

$$= \sum_{i,j} b_{\nu ij} a_{hij} = 0,$$

since F' and F are h apolar. Then $F' \sim F^* \sim G'$ gives $F' \sim G'$, where G' is any form h apolar to G. By symmetry the same result holds for linear transformations on z_j. The remainder of the argument is obvious and will be omitted.

Theorem 2 enables us to speak of apolar classes of forms $[F]$ and $[F']$. ($[F]$ denotes the class of forms equivalent to F.) Theorems 1 and 2, together with their analogs for i and j apolarity, can be applied to reduce the problem of classifying all trilinear forms to that of classifying trilinear forms in which no one of the rank invariants r_h, r_i, r_j is greater than half the product of the other two. To prove this we note first that no one of the r_ν can exceed the product of the other two.† Then it is obvious that one of r_h and $r'_h = r_i r_j - r_h$ does not exceed $r_i r_j / 2$.

An example will serve to illuminate such application. Let F have $r_h = 27$, $r_i = 10$, $r_j = 3$. Then if we take $m = 10$, $n = 3$, $[F]$ is h apolar to a unique class of forms $[F']$ with $r'_h = 3$, $r'_i \leqq 10$, $r'_j \leqq 3$. As $[F]$ runs through the classes of forms (27, 10, 3), there will be classes $[F']$ for each value of $r'_j \leqq 3$ and $r'_h / r'_j = 3/r'_j \leqq r'_i \leqq 3r'_j = r'_i r'_j$. Hence, given the classes of forms with $(r_h, r_i, r_j) = (3, 9, 3), (3, 8, 3), \cdots, (3, 1, 3), (3, 6, 2), (3, 5, 2), \cdots, (3, 2, 2), (3, 3, 1)$, we can at once write down the classes of forms (27, 10, 3).

But in the same way, the forms $(3, 9, 3), \cdots, (3, 5, 3)$ are i apolar to forms with $r_h \leqq 3$, $r_i \leqq 4$, $r_j \leqq 3$; and the forms (2, 6, 3), (2, 5, 3), (2, 4, 3) are apolar to forms with the r_ν all $\leqq 3$. Hence the classification of forms (27, 10, 3) with $810 = 3 \cdot 10 \cdot 27$ coefficients can be re-

† H. R. Brahana, *Metabelian groups and trilinear forms*, Duke Mathematical Journal, vol. 1 (1935), p. 191.

duced by apolarity considerations to the solution of linear equations and to the classification of forms in which all the r_ρ are less than or equal to three or $(r_h, r_i, r_j) = (3, 4, 3)$ and in which at most 36 coefficients appear.

3. **Pencils of bilinear forms.** Consider the trilinear form

$$G = \sum_{\rho=1}^{R} R_\rho + \sum_{\lambda=1}^{S} S_\lambda + T,$$

where

$$T = z_1 \sum_{\alpha,\beta=1}^{n} t_{\alpha\beta} x_\alpha y_\beta + z_2 \sum_{\alpha=1}^{n} x_\alpha y_\alpha, \quad R_\rho = \sum_{\alpha=1}^{m_\rho} (z_1 x_{\rho\alpha-1} + z_2 x_{\rho\alpha}) y_{\rho\alpha},$$

$$S_\lambda = \sum_{\alpha=1}^{n_\lambda} X_{\lambda\alpha}(z_1 Y_{\lambda\alpha-1} + z_2 Y_{\lambda\alpha}),$$

and all of the variables in the forms R_ρ, S_λ, T are linearly independent in the field K of G. We suppose also that $m_\rho \geqq m_{\rho+1}$, $n_\lambda \geqq n_{\lambda+1}$. The form G has two-way rank invariants

$$r_h = \sum_{\rho=1}^{R} (m_\rho + 1) + \sum_{\lambda=1}^{S} n_\lambda + n, \quad r_i = \sum_{\rho=1}^{R} m_\rho + \sum_{\lambda=1}^{S} (n_\lambda + 1) + n,$$

$$r_j = 2.$$

The form $\overline{G} = \sum_{\rho=1}^{\overline{R}} \overline{R}_\rho + \sum_{\lambda=1}^{\overline{S}} \overline{S}_\lambda + \overline{T}$ is h apolar to G (relative to r_i and 2),† where

$$\overline{T} = z_1 \sum_{\alpha=1}^{n} x_\alpha y_\alpha - z_2 \sum_{\alpha,\beta=1}^{n} t_{\beta\alpha} x_\alpha y_\beta,$$

$$\overline{R}_\rho = \sum_{\alpha=1}^{m_\rho - 1} x_{\rho\alpha}(z_1 y_{\rho\alpha+1} - z_2 y_{\cdot\alpha}),$$

$$\overline{S}_\lambda = \sum_{\alpha=1}^{n_\lambda + 1} Y_{\lambda\alpha-1}(z_1 X_{\lambda\alpha} - z_2 X_{\lambda\alpha-1}),$$

and the variables x and y are all independent. We have $\overline{S} = S$, $R - \overline{R} = j_1$, where j_α is the number of m_ρ equal to α.

If in G we let $z_1' = -z_2$, $z_2' = z_1$, $x_\alpha = y'_\alpha$, $x_{\rho\alpha} = y'_{\rho\alpha}$, $X_{\lambda\alpha} = Y'_{\lambda\alpha}$, $y_\alpha = x'_\alpha$, $y_{\rho\alpha} = x'_{\rho\alpha-1}$, $Y_{\lambda\alpha} = X_{\lambda\alpha-1}$, we obtain a form G' identical in form with G, having $T'(x, y) \equiv T(x, y)$ and the numbers $m_\rho' = m_\rho - 1$, $n_\lambda' = n_\lambda + 1$.

† That condition (1) for apolarity is satisfied is obvious if the two-way h-displays of G and $\overline{G}$ are written down. For (2) we note that $\bar{r}_h + r_h = (\sum(m_\rho - 1) + \sum(n_\lambda + 2) + n) + (\sum(m_\rho + 1) + \sum n_\lambda + n) = 2(\sum m_\rho + \sum(n_\lambda + 1) + n) = 2r_i$.

(We omit R_r' if $m_r'=0$.) Then $S'=S$, $R'=R-j_1$. Hence G' has two-way rank invariants

$$r_\lambda' = \sum_{r=1}^{R'} m_r + \sum_{\lambda=1}^{S} (n_\lambda + 1) + n,$$

$$r_i' = \sum_{r=1}^{R'} (m_r - 1) + \sum_{\lambda=1}^{S} (n_\lambda + 2) + n.$$

Considered as a pencil of bilinear forms with parameter $z=(z_1, z_2)$, G has minimal numbers m_r and n_λ and invariant factors (aside from 1) equal to those of T. Dickson* has proved that any pencil of bilinear forms F can be put in the canonical form G by non-singular (rational) transformations on the variables x, y, z. We have then $F\sim G$. Let $\overline{F}$ be h apolar to F (relative to r_i and 2). Then, by Theorem 2, $\overline{F}\sim\overline{G}$; and if F' is for $\overline{F}$ the analogue† of G' for $\overline{G}$, then $F'\sim G'$. This means in particular that F' has the same minimal numbers as G' and invariant factors projectively equivalent to those of F (since F, G, G', F' have, in turn, projectively equivalent invariant factors).

Let $F^{(\alpha)}$ be derived from $F^{(\alpha-1)}$ as F' is derived from $F^{(0)}=F$. We propose to calculate the minimal numbers m_r from the sequence of two-way rank invariants $r_\lambda^{(\alpha)}$, $r_i^{(\alpha)}$. Since these invariants are ranks of two-way matrices, we will then have the m_r expressed in terms of ranks of matrices, obtainable from F by solving linear equations.

To effect this calculation we shall evaluate the j_α defined above. By hypothesis m_1 is the biggest of the m_α, hence $j_\alpha=0$ for $\alpha>m_1$. Hence, as soon as we know the sequence j_α, $(\alpha=1, \cdots, m_1)$, we know all of the minimal numbers m_r. In particular $R^{(0)}=R=\sum_{\alpha=1}^{m_1} j_\alpha$, and $\sum_{r=1}^{R} m_r=\sum_{\alpha=1}^{m_1}\alpha j_\alpha$.

We saw above that $j_1=R^{(0)}-R^{(1)}$. Now

$$r_\lambda^{(0)} - r_\lambda^{(1)} = \left(\sum_{r=1}^{R^{(0)}} (m_r + 1) + \sum_{\lambda=1}^{S^{(0)}} n_\lambda + n\right) - \left(\sum_{r=1}^{R^{(0)}} m_r + \sum_{\lambda=1}^{S^{(0)}} (n_\lambda + 1) + n\right)$$

$$= [R^{(0)} + (R^{(0)} - R^{(1)}) - S^{(0)},$$

* Loc. cit., in particular Theorem 2, p. 248 and Theorem 3, p. 251. These results are stated for pairs of bilinear forms, but the numbers m_r, n_λ and the projective invariants of the invariant factors will constitute a complete set of invariants for pencils of bilinear forms. We use here this generalization of Dickson's result.

† That is, the matrix of F' is the transpose of that of F.

and similarly $r_i^{(0)} - r_i^{(1)} = R^{(0)} - S^{(0)}$. Hence $j_1 = R^{(0)} - R^{(1)} = (r_k^{(0)} - r_k^{(1)}) - (r_i^{(0)} - r_i^{(1)}) = (r_k^{(0)} - r_i^{(0)}) - (r_k^{(1)} - r_i^{(1)})$. By induction on α we show that $j_\alpha = (r_k^{(\alpha-1)} - r_i^{(\alpha-1)}) - (r_k^{(\alpha)} - r_i^{(\alpha)})$, which is the desired result.

The analogous process when we use i apolarity gives the numbers n_λ.

The following inequality gives an upper limit for m_1 and is of use in determining the number of steps the above procedure will require. If $M < m_1$, then $R > \sum_{\alpha=1}^{M} j_\alpha$ and

$$\sum_{\alpha=1}^{M} \alpha j_\alpha + (M+1) \leqq \sum_{\alpha=1}^{M} \alpha j_\alpha + m_1 \leqq \sum_{\alpha=1}^{m_1} \alpha j_\alpha = \sum_{\nu=1}^{R} m_\nu.$$

But

$$\sum_{\nu=1}^{R} m_\nu = r_k - n - \sum_{\lambda=1}^{S} n_\lambda - R \leqq r_k - n - R - S \leqq 2r_k - r_i - n - 2R.$$

($S = R + r_i - r_k$.) Hence $M < 2r_k - r_i - n - \sum_{\alpha=1}^{M} (\alpha+2) j_\alpha - 2$. Therefore m_1 is less than the smallest integer M for which this inequality does not hold. Or, if one computes the rank, $r_k - R = r_i - S$, of the characteristic matrix $M_z(z)$ of F, the value of R thus obtained, together with $R = \sum_{\alpha=1}^{m_1} j_\alpha$, shows one exactly when enough steps have been taken.

Dickson's process* for determining m_ν, n_λ requires $R+S$ steps, each of which involves solving several sets of linear equations, and finding the minimum degree $m_{R-\alpha}$ in z of the forms $A_k^{(\alpha)}$ such that $\sum A_k^{(\alpha)} F_k^{(\alpha)} \equiv 0$ in z and y, where $F_k^{(\alpha)}$ is the coefficient of x_k in $F^{(\alpha)}$ (the form obtained by the $(\alpha-1)$st step). No method, other than trial and error, is given for determining $m_{R-\alpha}$.

The apolarity process requires $m_1 + n_1$ steps, each of which involves solving one set of linear equations and evaluating the rank of a matrix (which is also the first step in solving the next set of linear equations). This gives, at the αth step, a specific formula for the number of m_ν equal to α (which is the chief value of the method).

Hence, if $R+S$ is small, Dickson's method is the shorter, but if $m_1 + n_1 < R + S$, or if $R+S$ and $m_1 + n_1$ are large, the apolarity method is shorter.

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* Loc. cit.

ON SOME NEW CONGRUENCES IN THE THEORY OF BERNOULLI'S NUMBERS

N. G. W. H. BEEGER

For Bernoulli's numbers the following relations are known:

$$(h+1)^n = h^n; \qquad n > 1, \qquad h_1 = -\tfrac{1}{2}, \qquad B_n = (-1)^{n-1}h_{2n};$$
$$h_{2n+1} = 0 \quad \text{for} \quad n > 0.$$

For the symbol $k^n = h^{n+1}/(n+1)$ Kummer proved the congruence

$$k^a(1-k^b)^c \equiv 0 \pmod{(p^e, p^{ec})}, \tag{1}$$

p being a prime, $b = p^{e-1}(p-1)b_1$, $a+1 \not\equiv 0 \pmod{(p-1)}$. G. Frobenius* has given another proof of this congruence, without using infinite series. I shall now prove the congruence

$$(-1)^{i-1}k^{a+mb} \equiv \sum_{s=1}^{i} (-1)^{s-1}C_{m,s-1}C_{m-s,i-s}k^{a+(s-1)b} \pmod{p^i}, \tag{2}$$
$$b = p-1,$$

which is equivalent to

$$(-1)^{i-1}\frac{B_{n+m\mu}}{2n+2m\mu} \equiv \sum_{s=1}^{i} (-1)^{s-1+(m-s+1)\mu} \cdot C_{m,s-1}C_{m-s,i-s}\frac{B_{n+(s-1)\mu}}{2n+2(s-1)\mu} \pmod{p^i}, \tag{3}$$

$C_{m,0} = 1$, $m \geqq i$, $i < 2n-1$, $2n \not\equiv 0 \pmod{(p-1)}$, $\mu = (p-1)/2$.

Take, in (1), $b = p-1$, $c = i$, $a = 2n-1$; then (1) gives

$$(-1)^{i-1}k^{a+bi} \equiv \sum_{s=1}^{i} (-1)^{s-1}C_{i,s-1}k^{a+(s-1)b} \pmod{p^i}.$$

Hence (2) is proved for the case $m = i$. Now suppose that (2) is proved for $m = i, i+1, i+2, \cdots, m$. From (1) it follows that

$$(-1)^m k^{a+(m+1)b} \equiv \sum_{s=1}^{i} (-1)^{s-1}C_{m+1,s-1}k^{a+(s-1)b} + \sum_{s=i+1}^{m+1} (-1)^{s-1}C_{m+1,s-1}k^{a+(s-1)b} \pmod{p^{m+1}}. \tag{4}$$

* Sitzungsberichte der Preussischen Akademie, vol. 39 (1910), p. 809

By substituting, for each term of the second sum in the right-hand side of (4), the series from (2), we obtain

$$(5)\qquad \begin{aligned}(-1)^m k^{a+(m+1)b} \equiv \sum_{s=1}^{i} (-1)^{s-1} k^{a+(s-1)b} (&C_{m+1,s-1} \\ &- C_{m+1,s}C_{i,s-1}C_{i-s,i-s} + C_{m+1,i+1}C_{i+1,s-1}C_{i-s+1,i-s} \\ &- \cdots \pm C_{m+1,m}C_{m,s-1}C_{m-s,i-s}) \pmod{p^i}.\end{aligned}$$

Let the coefficient of k^a be denoted by S_m. Then

$$(6)\qquad \begin{aligned}S_m = 1 &- C_{m+1,i}C_{i-1,i-1} + C_{m+1,i+1}C_{i,i-1} \\ &- \cdots + (-1)^{m+i-1}C_{m+1,m}C_{m-1,i-1},\end{aligned}$$

and, using the known relation

$$C_{m+1,c} - C_{m,c} = C_{m,c-1},$$

we have

$$(7)\qquad \begin{aligned}S_m - S_{m-1} &= \sum_{j=i}^{m-1} (-1)^{j-i+1} C_{m,j-1}C_{j-1,i-1} + (-1)^{m+i-1}C_{m+1,m}C_{m-1,i-1} \\ &= C_{m,i-1} \sum_{j=i}^{m-1} (-1)^{j-i+1} C_{m-i+1,m-j+1} \\ &\quad + (-1)^{m+i-1}C_{m+1,m}C_{m-1,i-1} \\ &= (-1)^{m-i+1}(C_{m,i-1} + C_{m-1,i-1}).\end{aligned}$$

From (6) it follows that $S_i = -i$; hence from (7) we have

$$\begin{aligned}S_m - S_i &= C_{i,i-1} - C_{i+1,i-1} + \cdots + (-1)^{m-i+1} C_{m-1,i-1} \\ &\quad + (C_{i+1,i-1} - C_{i+2,i-1} + \cdots + (-1)^{m-i+1}C_{m,i-1}),\end{aligned}$$

$$(8)\qquad S_m = (-1)^{m-i+1}C_{m,i-1}.$$

Let further the coefficient of k^{a+jb} in the second member of (5) be denoted by S_m'; then

$$\begin{aligned}S_m' &= C_{m+1,j} - C_{m+1,j}C_{i,j}C_{i-j-1,i-j-1} + \cdots \pm C_{m+1,m}C_{m,j}C_{m-j-1,i-j-1} \\ &= C_{m+1,j}(1 - C_{m-j+1,i-j}C_{i-j-1,i-j-1} + C_{m-j+1,i-j+1}C_{i-j,i-j-1} \\ &\quad - \cdots \pm C_{m-j+1,m-j}C_{m-j-1,i-j-1});\end{aligned}$$

hence

$$S_m' = C_{m+1,j}S_{m-j} = C_{m+1,j}C_{m-j,i-j-1}(-1)^{m-i+1}$$

by (8). From (5) the congruence (2) is now proved for $(m+1)$; hence (2) is true in general for all numbers $m = i, i+1, \cdots$.

In order to get a congruence analogous to (2) and (3), but for a modulus which is a higher power of p than p^a, take, in (2), $a+jb$ in place of a, and replace $(m+j)$ by m. We have

$$(-1)^{i-1}k^{a+mb} \equiv \sum_{s=1}^{i} (-1)^{s-1} C_{m-j,s-1} C_{m-j-s,i-s} k^{a+(s+j-1)b} \pmod{p^i}, \tag{9}$$
$$m \geqq i+j,\ i \leqq a+jb,\ a+1 \not\equiv 0 \pmod{(p-1)},$$

with the equivalent relation

$$(-1)^{i-1}\frac{B_{n+m\mu}}{2n+2m\mu} \equiv \sum_{s=1}^{i} (-1)^{s-1+(m-s+1)\mu} \cdot C_{m-j,s-1} C_{m-j-s,i-s} \frac{B_{n+(s+j-1)\mu}}{2n+2\mu(s+j-1)} \pmod{p^i}, \tag{10}$$
$$m \geqq i+j,\ i \leqq 2n-1+j(p-1),\ 2n \not\equiv 0 \pmod{(p-1)}.$$

A prime $p>3$ is said to be irregular if it divides one of the numbers $B_1, B_2, \cdots, B_{\mu-1}$, say B_n. It is known by (1) that in this case each number $B_{n+m\mu}$ is divisible by p.

THEOREM. *If p is an irregular prime, and if $k^a \equiv 0 \pmod{p}$, then for each number i the positive integers $m_1, m_2, \cdots, m_{i-1} < p$, can be determined uniquely by the chain of congruences*

$$k^a \equiv m_1 P \pmod{p^2},$$
$$k^{a+m_1 b} \equiv m_2 p P \pmod{p^3},$$
$$k^{a+(m_1+m_2 p)b} \equiv m_3 p^2 P \pmod{p^4},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$k^{a+(m_1+m_2 p+\cdots+m_{i-2}p^{i-2})b} \equiv m_{i-1} p^{i-2} P \pmod{p^i},$$

provided that

$$P = k^a - k^{a+b} \not\equiv 0 \pmod{p^2};$$

consequently

$$k^{a+(m_1+m_2 p+\cdots+m_{i-1}p^{i-2})b} \equiv 0 \pmod{p^i}.$$

In the above k is defined as at the beginning of the article, and $b=p-1$.

PROOF. In the congruence (2) take $i=2$; this gives

$$-k^{a+mb} \equiv (m-1)k^a - mk^{a+b} \pmod{p^2}, \tag{11}$$
$$k^{a+mb} \equiv k^a - m(k^a - k^{a+b}) \pmod{p^2}.$$

The congruence

$$k^a - m(k^a - k^{a+b}) \equiv 0 \pmod{p^2},$$

wherein k^a and k^{a+b} are divisible by p, has one solution $m_1 < p$ if and only if $P = k^a - k^{a+b} \not\equiv 0 \pmod{p^2}$, and it follows from (11) that

$$k^{a+m_1 b} \equiv 0 \pmod{p^2}.$$

Hence the theorem is proved for $i = 2$. Suppose that it is proved for $2, 3, 4, \cdots, i$, and put

$$m_1 + m_2 p + \cdots + m_{i-2} p^{i-3} = m';$$

then $k^{a+m'b} \equiv 0 \pmod{p^i}$. Now take the congruence (9) for $(i+1)$ in place of i, which gives

$$(-1)^i k^{a+mb} \equiv \sum_{s=1}^{i+1} (-1)^{s-1} C_{m-j,s-1} C_{m-j-s,i+1-s} k^{a+(s+j-1)b} \pmod{p^{i+1}}.$$

Let the polynomial in the right-hand side be denoted by $G(m)$. Then

$$G(m' + p^{i-1}x) \equiv G(m') + p^{i-1} x G'(m') \pmod{p^{i+1}}; \tag{11a}$$

also

$$G(m') \equiv (-1)^i k^{a+m'b} \pmod{p^{i+1}}$$

from the definition of G, and (2) gives, for $i = 2$,

$$G(m) = (-1)^i k^{a+mb} \equiv (-1)^{i-1} \left\{ (m-1)k^a - m k^{a+b} \right\} \pmod{p^2};$$

hence

$$G'(m) \equiv (-1)^{i-1} \left\{ k^a - k^{a+b} \right\} \pmod{p^2},$$

and setting $m = m'$ in this relation, we get from (11a)

$$G(m' + p^{i-1}x) \equiv (-1)^i k^{a+m'b} + (-1)^{i-1} p^{i-1} x (k^a - k^{a+b}) \pmod{p^{i+1}},$$

and hence

$$(-1)^i k^{a+(m'+p^{i-1}x)b} \equiv (-1)^i k^{a+m'b} + (-1)^{i-1} p^{i-1} x (k^a - k^{a+b}) \pmod{p^{i+1}}.$$

Now $k^{a+m'b} \equiv 0 \pmod{p^i}$. The congruence

$$k^{a+m'b} - p^{i-1} x (k^a - k^{a+b}) \equiv 0 \pmod{p^{i+1}}$$

has therefore one solution $x = m_i < p$ if and only if $k^a - k^{a+b} \not\equiv 0 \pmod{p^2}$, and then $k^{a+(m'+p^{i-1}m_i)b} \equiv 0 \pmod{p^{i+1}}$. The theorem is proved for $(i+1)$ and hence is true for all values of i.

It follows immediately from this theorem that for each number i, as large as we please, the numbers $m_1, m_2, \cdots, m_{i-1}$ can be deter-

mined so that if $B_n \equiv 0 \pmod{p}$, $n < p$, p an irregular prime, then

$$B_{n+(m_1+m_2p+\cdots+m_{i-1}p^{i-2})\mu} \equiv 0 \pmod{p^i},$$

if

$$\frac{B_n}{2n} \not\equiv (-1)^{(p-1)/2} \frac{B_{n+(p-1)/2}}{2n+p-1}.$$

Pollaczek* calculated, in the cases $n=16$, $p=37$; $n=22$, $p=59$; and $n=29$, $p=67$, the number m_1 for which $B_{n+m_1\mu} \equiv 0 \pmod{p^2}$. His calculations gave me the idea to construct my congruences (3) and (10) and to formulate the theorem.

The substitution of $m=2n$ in (3) gives the result

$$(12) \qquad (-1)^{i-1}\frac{B_{np}}{np} \equiv \sum_{s=1}^{i} (-1)^{(s-1)(\mu+1)} C_{2n,s-1} C_{2n-s,i-s} \cdot \frac{B_{n+(s-1)\mu}}{2n+2(s-1)\mu} \pmod{p^i}, \quad 2n \not\equiv 0 \pmod{(p-1)}.$$

The case $i=2$ is of special interest. H. S. Vandiver and his collaborators, in their researches about the second case of Fermat's last theorem,† have made very extensive calculations to find the residues of B_{np}, modulo p^3, p being an irregular prime less than 211, not knowing the congruence (12). For $B_n \equiv 0 \pmod{p}$, we have $n < \mu$; then $B_{np} \equiv 0 \pmod{p^2}$, and (12) gives, for $i=2$,

$$B_{np} \equiv -\frac{p}{2n-1}\left\{(2n-1)^2 B_n - (-1)^{(p-1)/2}(2n)^2 B_{n+\mu}\right\} \pmod{p^3}.$$

Using the existing tables of Bernoulli's numbers we can obtain from this congruence the residue of B_{np}, modulo p^3, after a simple calculation. Thus I have checked the results of Vandiver (except for $p=157$ and 149; B_{133} and B_{139} not being in the tables) and have found them all correct.

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* Mathematische Zeitschrift, vol. 21 (1924), pp. 28–31. Some of his results are wrong. They should be $B_{22} \equiv 50 \cdot 59$, $B_{51} \equiv 42 \cdot 59 \pmod{59^2}$, $B_{62} \equiv 37 \cdot 67 \pmod{67^2}$.

† Transactions of this Society, vol. 31 (1929), pp. 613, 639–642.

CONCERNING R. L. MOORE'S AXIOM 5*

F. B. JONES

In Axiom 5† of his *Foundations of Point Set Theory*, R. L. Moore postulates the existence of a simple closed curve having certain properties. Professor Moore pointed out to me, while I was still in one of his classes, that in some respects this was undesirable, and that if possible it would be better to state a simple axiom without this feature from which Axiom 5 would follow as a theorem in the presence of the other axioms assumed up to this point in *Foundations*, namely, Axioms 0–4. It is the object of this paper to show that this is possible.

Suppose that S, the set of all points, is a space satisfying Axioms 0–4 of Moore's *Foundations* and the following axiom:

AXIOM 5′. *If A is a point of a region R and B is a point distinct from A, there exists in R a compact continuum separating A from B.*

THEOREM 1. *If P is a point of a connected domain D, then $D-P$ is connected.*

PROOF. Suppose that $D-P$ is not connected. There exist two distinct components H and K of $D-P$, each having P on its boundary. Let A and B denote points of H and K, respectively, and let AB denote an arc from A to B in $S-P$.‡ Let R denote a region lying in D, containing P but containing no point of AB. By Axiom 5′, R contains a continuum T separating A from P. The continuum T must, therefore, contain a point of H and a point of K. Since T is a connected subset of $D-P$, this involves a contradiction.

THEOREM 2. *The space S contains no local cut points.*

THEOREM 3. *If A and B are distinct points, there exists a simple closed curve containing A and B.*§

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* Presented to the Society, April 9, 1938.

† *If A is a point of a region R and B is a point distinct from A, there exists, in R, a simple closed curve separating A from B.* Axioms, theorems, and definitions used, but not explicitly stated in this paper, are those of R. L. Moore's *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. This book will be referred to as *Foundations*.

‡ The existence of AB follows from Theorem 1 of Chapter 2 and Axiom 3 of *Foundations*.

§ For an argument to prove Theorem 3 using Theorem 2, the reader is referred to certain portions of the argument for Theorem A on page 54 of the author's paper, *Concerning certain topologically flat spaces*, Transactions of this Society, vol. 42 (1937), pp. 53–93.

THEOREM 4. *If A and B are distinct points, there exists a simple closed curve separating A from B.**

THEOREM 5. *If P is a point not belonging to the closed and compact point set M, there exists a simple closed curve separating P from M.*†

THEOREM 6. *If the points A and B belong to different components of the closed and compact point set M, there exists a simple closed curve separating A from B and containing no point of M.*‡

THEOREM 7. *If the points A and B are separated from each other by the closed and compact point set M, then they are separated from each other by a continuum which is a subset of M and which contains no proper subset that separates A from B.*§

THEOREM 8. *If M is a compact continuum lying in a domain D and N is a compact continuum containing no point of M, then there exists in D a compact continuum L which separates M from N.*

PROOF. Let B denote a point of N. For each point A of M there exists a region R containing A and lying in $D-D\cdot N$. By Axiom 5′ there exists in R a compact continuum T separating A from B. Let Q denote the component of $S-T$ containing A, and let G denote the collection of all such domains. Since G covers M, there exists a finite subcollection G' of G covering M. Then the boundary β of the domain obtained by adding together the elements of G' is a compact subset of $D-D\cdot N$ which separates M from B. It follows from Theorem 7 that β contains a continuum L separating M from B and, consequently, from N.

THEOREM 9. *No arc separates S.*

PROOF. Suppose, on the contrary, that there exists an arc separating a point A from a point B. By Theorem 7, this arc contains a continuum M separating A from B, such that no proper subset of M separates A from B. By Axiom 3, M is non-degenerate. Hence M is an arc EF. Let ω denote an interior point of EF. Neither the arc $E\omega$ nor the arc $F\omega$ of EF separates A from B. Hence there exist two arcs AOB and APB from A to B such that (1) $AOB\cdot F\omega=0$, (2) $APB\cdot E\omega=0$, (3) $AOB\cdot E\omega$ is an arc T_O, and O is the first point

* Loc. cit., p. 57 for a proof of this theorem.

† Loc. cit., p. 58 for a proof of this theorem.

‡ Theorem 6 may be proved by using Theorem 5 and a slight modification of R. L. Moore's argument for Theorem 10 on page 185 of *Foundations*.

§ For a proof of Theorem 7 see the proof of Theorem 24 on page 194 of *Foundations*. It is valid without modification for the space considered here.

of $E\omega$ in T_O, (4) $APB \cdot F\omega$ is an arc T_P, and P is the first point of $F\omega$ in T_P, and (5) $AOB \cdot APB$ is two arcs AA' and BB'.* There exists in $AOB+APB$ a simple closed curve $A'OB'PA'$ whose common part with EF is T_O+T_P. Let J denote this simple closed curve, and let D denote the interior of J with respect to ω as "the point at infinity." If D contains neither E nor F, then it is evident that EF does not separate A from B. On the other hand, suppose that E is in D. By Theorem 6, there exists a simple closed curve C separating ω from O and containing no point of the closed and compact point set $F\omega+EO$ (of $E\omega$) $+AOB$. It is clear that both C and J are bounded, and that C encloses the point O of J but encloses neither P nor, consequently, the interior of J.† By Theorem 11 on page 163 of *Foundations*, there exists a simple closed curve Q such that (1) Q is a subset of $C+J$, (2) the interior of Q is a subset both of D and of I, the interior of C, and (3) Q contains a segment T of C which lies wholly in D but whose extremities separate O from P on J. By (1) and (2), $\overline{T}$ contains no point of EF not in T_O, and by (3), $\overline{T}$ contains no point of T_O. Hence $T+APB$ contains an arc from A to B not intersecting EF, which is a contradiction.

THEOREM 10. *If AB is an arc lying in a connected domain D, then $D-AB$ is connected.*

PROOF. Suppose the contrary. Then there exist two distinct components H and K of $D-AB$, each having boundary points in AB. Let X and Y denote points of H and K, respectively. Since, by Theorem 9, AB does not separate space, there exists in $S-AB$ an arc XY from X to Y. By Theorem 8, D contains a continuum L separating AB from XY. Hence L contains both a point of H and a point of K, which is a contradiction.

THEOREM 11. *If a connected domain D is regarded as a space in which the term "region" is interpreted to mean a connected open subset of D, then with respect to this interpretation of "point" and "region," Axioms 0–4 and 5′ are satisfied, and "limit point" is invariant under this change.*

PROOF. By Theorem 9 on page 96 of *Foundations*, Axioms 0–2 are satisfied. Theorem 1 of this paper shows that Axiom 3 is satisfied, and it is evident that Axiom 5′ is satisfied. It remains to be shown that Axiom 4 is satisfied.

* Of course, either or both of the arcs AA' and BB' may be degenerate. In any case the argument is unaffected.

† This portion of the argument will likely be confusing if the reader does not have in mind the convention and definitions stated on page 153 of *Foundations*.

Let J denote a simple closed curve lying in D. From Theorem 10 it follows that every component of $D-J$ has J for its complete boundary with respect to D. It follows from Axiom 4 that there are at least two such components. Suppose that there are more than two. Let H and K denote two of these components which lie in the same component of $S-J$. Then there exists an arc T not intersecting J which contains a point of H and a point of K. By Theorem 8, there exists in D a continuum separating J from T in S. This continuum must, therefore, contain both a point of H and a point of K, which is a contradiction.

THEOREM 12. *If A is a point of a region R, and B is a point distinct from A, there exists in R a simple closed curve separating A from B.*

PROOF. Let D denote a connected domain continuing A and lying in R. By Theorem 8, there exists a continuum L lying in D and separating A from B. It follows from Theorems 5 and 11 that there exists in D a simple curve J such that $D-J$ is the sum of two mutually exclusive connected domains I and E both having J on their boundaries, with I containing A and E containing T. But I and E must lie in different components of $S-J$ since $I+E=D-J$. Hence J separates A from T and, consequently, from B in S.

THEOREM 13. *A space satisfying Axioms* 0–4 *and* 5′ *satisfies Axioms* 0–5.

That Theorem 13 does not remain true, if Axiom 5′ is weakened by omitting the compactness condition on the separating continuum, may be seen by considering the space consisting of all the points of the number plane with positive ordinates together with the origin. If "limit point" is interpreted in the ordinary sense, it is easy to see that Axiom 5 does not hold at the origin.

Likewise, Theorem 13 does not remain true if Axiom 5′ is weakened by replacing the words "*a compact continuum*" by "*a closed and compact point set.*" An example to show this is included in a paper by the author which is as yet unpublished. *

It may be of interest to the reader that there exists a space satisfying Axioms 0–5 which, even though metric and completely separable, is, nevertheless, not homeomorphic with a subset of a plane or a sphere.†

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* *Certain equivalences and subsets of a plane.* Presented to the Society, June 18, 1936. For the abstract see this Bulletin, vol. 42 (1936), p. 484.

† An example of such a space has been discovered by R. L. Moore and, although unpublished, has been well known among his students for some five or six years.

ENUMERATIVE PROPERTIES OF PLANE CONNECTED n-LINES

B. C. WONG

1. **Introduction.** Consider n distinct lines $a_1, a_2, \cdots, a_n$, no two of which are parallel, in a euclidean plane ϕ. These n lines, together with their $n(n-1)/2$ points of intersection, some or all of which may be coincident, form a configuration which we temporarily denote by K. Any one of the points in K, say the point of intersection of the lines a_i and a_j, may be regarded as a virtually non-present intersection of a_i and a_j. Such a point will be called a point of non-connection and will be denoted by Q_{ij}. The lines a_i and a_j are then said to be disconnected or to have virtually no intersection at Q_{ij}. Let d be the number of points of non-connection in K, where

$$0 \leqq d \leqq (n-1)(n-2)/2. \tag{1}$$

The condition expressed by (1) will be explained in §2. Any point of K, not regarded as a point of non-connection, will be called a point of connection and will be denoted by P_{ij} if it is the intersection of the lines a_i and a_j. If the d points of non-connection in K are taken in such a way that each of the n lines has one point of connection with at least one of the remaining lines, the resulting configuration is called a plane connected n-line with d points of non-connection and will henceforth be denoted by γ_d^n or just γ.

The object of this paper is to derive some of the enumerative properties of γ. What these properties are will be explained as we proceed. They will all be expressed in terms of n and d.

2. **The maximum number of points of non-connection.** If $d=0$, then all the $n(n-1)/2$ points in γ_0 are points of connection. We may call γ_0^n an absolutely connected n-line. Suppose $d>0$. Obviously no $n-1$ of the d assumed points of non-connection can be on any one line, say a_1. For, if $n-1$ of them did lie on a_1, then a_1 would be disconnected from the remaining lines, and γ would not be a connected n-line. Let there be $n-2$ such points on a_1. Then a_1 is connected with another line, say a_2, by the point P_{12}. If a_2 is to be connected with a third line a_3, then no more than $n-3$ of the remaining points of non-connection can be on a_2. Similarly, if a_3 is to be connected with a fourth line a_4, then a_3 cannot have on it more than $n-4$ of the remaining $d-(n-2)-(n-3)$ points of non-connection. Continuing this proc-

ess, we find that d cannot exceed the number

$$D = (n-2) + (n-3) + \cdots + 1 = (n-1)(n-2)/2. \tag{2}$$

If $d=D$, we say that γ_D^n is an open n-line; and if $d>D$, γ_d^n is not a single connected n-line and is said to be composite. Unless the contrary is stated, we shall suppose that $0 \leqq d \leqq D$, and that no more than $\nu(2n-\nu-3)/2$, which is the sum of the first ν terms in (2), of the d assumed points of non-connection can be on any ν of the n lines of γ_d^n.

The maximum value D of d may be obtained in other ways. For example, suppose an n-line such that each of the points in which one of the n lines meets the remaining $n-1$ lines is a point of connection and that each of the points in which the $n-1$ remaining lines meet, two by two, is a point of non-connection. Then the total number of points of non-connection is the same as that given by (2).

The number of points of connection in a non-composite γ_d^n is always

$$M = n(n-1)/2 - d. \tag{3}$$

For an open n-line, that is, for the case $d=D$, we have $M=n-1$.

3. **Adjacent and non-adjacent points of γ.** Two points of connection, p_{ij}, p_{ik}, on the same line a_i, are said to be adjacent if there is no third point of connection, P_{il}, on a_i separating them. If they are separated by one or more points of connection on a_i, they are said to be non-adjacent. Two points of connection, such as P_{ij}, P_{kl}, which are not on any one line, are said to be opposite; they are also non-adjacent.

A point P_{ij} (or a point Q_{ij}) which is adjacent to four points of connection, two on a_i and two on a_j, is called an inner point of connection (or of non-connection). If it is adjacent to less than four points of connection, it is called an outer point of connection (or of non-connection).

4. **Transformation of n-lines.** We assume that a given n-line, in which we may suppose that no three or more lines are concurrent, may be transformed into another by the following process: changing some or all of the d points of non-connection in the given n-line into points of connection and an equal number of points of connection into points of non-connection so that the number d of points of non-connection is preserved. In particular, we may transform any given n-line in which some or all of the d points of non-connection are inner points into another in which all the d points of non-connection are

outer points. For purposes of enumeration we shall use the transformed n-line.

5. **Segments.** The portion of any line in γ_d^n included between two adjacent points of connection is called a closed segment belonging to γ_d^n. The term "closed segment" may sometimes be replaced by "pair of adjacent points of connection." That portion of a line issuing from a point of connection and extending indefinitely without passing through another point of connection may be called an open segment.

The number of closed segments in γ_d^n can be easily obtained. First, let us consider an absolutely connected n-line γ_0^n. Since there are $n-1$ points of connection on each line of γ_0^n and therefore $n-2$ closed segments, the total number of closed segments is $n(n-2)$. Suppose now we remove one of the points of connection, say P_{ij}, that is, change it into a point Q_{ij} of non-connection. If P_{ij} is an inner point, it is common to four closed segments, two on a_i and two on a_j. After the removal or change, the four closed segments coalesce into two. If it is an outer point, it may be common to two closed segments, one on a_i and one on a_j, and, after its removal, these two segments are no longer closed; or it may be common to three closed segments, one on a_i (or a_j) and two on a_j (or a_i) and, after its removal, the one segment becomes open and the other two coalesce into one. In all these three possible cases, the removal of one point of connection diminishes the number of closed segments by two. Therefore, the number of closed segments or pairs of adjacent points of connection in γ_d^n is

$$I = n(n-2) - 2d. \tag{4}$$

6. **Pairs of non-adjacent points of connection.** We now calculate the number T of pairs of non-adjacent points of connection in γ_d^n. There are $M(M-1)/2$, where M is given by (3), pairs of points, both adjacent and non-adjacent, of connection; and if we subtract from this the number I, given by (4), of pairs of adjacent points of non-connection, we have the desired number

$$\begin{aligned} T &= M(M-1)/2 - I \\ &= n(n-2)(n^2-9)/8 - d(n+2)(n-3)/2 + d(d-1)/2. \end{aligned} \tag{5}$$

7. **Circuits.** Let us call a closed circuit or just a circuit any simple closed polygon formed by three or more adjacent segments of γ^n. On a circuit the number of points of connection is equal to the number of segments. An open n-line γ_D^n has no circuit. An absolutely connected 3-line $\gamma_0^3 \equiv a_1a_2a_3$ has one circuit, and a 4-line $\gamma_0^4 \equiv a_1a_2a_3a_4$ has three. By adding lines $a_5, a_6, \cdots, a_n$ successively, we find that the number

of circuits in the resulting n-line γ_0^n is

$$P = 1 + 2 + 3 + \cdots + (n-2) = (n-1)(n-2)/2 = D. \tag{6}$$

This is the maximum number of circuits any connected n-line can have.

Now we remove d points of connection from γ_0^n. If in the resulting n-line some or all of the d points of non-connection are inner points, we transform the n-line into another one γ_d^n in which all the d points of non-connection are outer points. It is easy to see that the removal of each point of connection from the original γ_0^n diminishes the number of circuits by one. Therefore, the number p of circuits in γ_d^n is such that

$$p + d = P = D = (n-1)(n-2)/2. \tag{7}$$

8. **Relations to plane curves.** When an n-line γ_d^n is given, n and d are assumed known. The elementary enumerative properties of γ_d^n so far obtained are all expressed in terms of n and d. By means of (7) M, I, T may be expressed as follows:

$$M = n - 1 + p,$$
$$I = n - 2 + 2p,$$
$$T = (n-3)(n-2+2p)/2 + p(p-1)/2.$$

It is of interest to note that these formulas or those expressed in terms of n and d bear a close relation to Plucker's equations connecting the class m, the number i of inflexions, the number t of bitangents, and also the genus p of a plane algebraic curve of order n with d nodes and no cusps. Comparing our results with those equations, we find that $m=2M$, $i=3I$, $t=4T$, and that the genus of the curve is equal to the number of circuits in γ_d^n. At present we are not offering any explanation* for these phenomena.

9. **Conclusion.** Our results are not entirely new. A number of geometers† have already derived numerous enumerative properties of algebraic curves in r-space from those of n-lines also in r-space. The writer himself has in two papers‡ done practically the same thing in

* An explanation may be offered from the point of view of topology but at present we are still in search of a more elementary one.

† Severi, Zeuthen, Tanturri, for example. See C. Segre, *Mehrdimensionale Räume*, Encyklopädie der mathematischen Wissenschaften, vol. III$_2$, no. 7, pp. 878–890.

‡ *On loci of* $(r-2)$*-spaces incident with curves in* r*-space*, this Bulletin, vol. 36 (1930), pp. 755–761, and *Enumerative properties of* r*-space curves*, this Bulletin, vol. 40 (1934), pp. 291–296.

a different manner, but has not explained sufficiently the method of deriving the enumerative properties of the n-lines and that of translating them into those of algebraic curves. The present paper is an attempt to put this method on a firmer basis, attention being confined to plane n-lines only.

Again, Severi* has considered n-lines for the purpose of classifying algebraic curves, and has also discussed the question as to whether a family of curves of order n defined in a certain manner contains at least one n-line as one of its members. Our purpose here has nothing to do with Severi's problems. For us, two n-lines, however different they may be according to Severi or from the point of view of collineations, have the same enumerative properties, provided they have the same number of points of non-connection or the same number of circuits, and our purpose is to obtain these properties and incidentally point out their relations to those of curves of order n with d nodes in the plane.

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* Severi-Löffler, *Vorlesungen über algebraische Geometrie*, 1921, Appendix G.

A RELATIVE OF THE LEMMA OF SCHWARZ*

E. F. BECKENBACH

1. **Introduction.** Let $w=f(z)$, where $z=u+iv$, be analytic for $|z|<1$, and let

$$d(r, \theta; f') = |f(re^{i\theta}) - f(0)| = \left| \int_0^r f'(\rho e^{i\theta}) d\rho \right|,$$

so that $d(r, \theta; f')$ is the length of the segment on the w-plane between the image of the point $z=0$ and the image of the point $z=re^{i\theta}$.

The lemma of Schwarz is the following:

THEOREM 1. *Let $w=f(z)$ be analytic for $|z|<1$. If*

$$d(r, \theta; f') \leqq 1$$

for all (r, θ) with $r<1$, then

$$d(r, \theta; f') \leqq r \tag{1}$$

and

$$|f'(0)| \leqq 1. \tag{2}$$

The sign of equality holds in (1) (for $r \neq 0$) and in (2), if and only if $|f'(z)| \equiv 1$; that is, if and only if the transformation $w=f(z)$ is a rigid motion.

If the (real) function $g(z)$ is subharmonic for $|z|<1$, then the Lebesgue integral

$$l(r, \theta; g) = \int_0^r g(\rho e^{i\theta}) d\rho$$

actually exists. We shall prove the following theorem:

THEOREM 2. *Let $g(z)$ be subharmonic for $|z|<1$. If*

$$l(r, \theta; g) \leqq 1 \tag{3}$$

for all (r, θ) with $r<1$, then

$$l(r, \theta; g) \leqq r \tag{4}$$

and

$$g(0) \leqq 1. \tag{5}$$

* Presented to the Society, December 28, 1937.

The sign of equality holds in (4) (*for* $r \neq 0$) *and in* (5), *if and only if* $g(z) \equiv 1$.

In particular, for the function $g(z) \equiv |f'(z)|$, where $f(z)$ is analytic for $|z| < 1$,

$$l(r, \theta; |f'|) = \int_0^r |f'(\rho e^{i\theta})| \, d\rho$$

is the length of the image on the w-plane of the segment between the points $z = 0$ and $z = re^{i\theta}$. The following theorem, which is a special case of Theorem 2, appears to be rather closely related to the lemma of Schwarz:

THEOREM 3. *Let* $w = f(z)$ *be analytic for* $|z| < 1$. *If*

$$l(r, \theta; |f'|) \leqq 1$$

for all (r, θ) *with* $r < 1$, *then*

$$l(r, \theta; |f'|) \leqq r \tag{6}$$

and

$$|f'(0)| \leqq 1. \tag{7}$$

The sign of equality holds in (6) (*for* $r \neq 0$) *and in* (7), *if and only if* $|f'(z)| \equiv 1$; *that is, if and only if the transformation* $w = f(z)$ *is a rigid motion.*

Let the real functions

$$x_j = x_j(z) = x_j(u + iv), \qquad j = 1, 2, 3, \tag{8}$$

have continuous derivatives of the third order with respect to u, v, in $|z| < 1$, with

$$E = G = [\lambda(z)]^2, \qquad F = 0, \tag{9}$$

where E, F, G are the fundamental quantities of the first order, so that the functions (8) give an isothermic map of $|z| < 1$ on a surface S, that is, a map which is conformal except at points where $\lambda = 0$.

The Gaussian curvature K is defined on S except at points where $\lambda = 0$.

The function

$$l(r, \theta; \lambda) = \int_0^r \lambda(\rho e^{i\theta}) d\rho$$

denotes now the length of the image on S of the segment between the

points $z=0$ and $z=re^{i\theta}$. We have the following further special case of Theorem 2:

THEOREM 4. *Let the Gaussian curvature K be less than or equal to zero wherever K is defined on the above surface S. If*

$$l(r, \theta; \lambda) \leqq 1$$

for all (r, θ) with $r<1$, then

$$l(r, \theta; \lambda) \leqq r \tag{10}$$

and

$$\lambda(0) \leqq 1. \tag{11}$$

The sign of equality holds in (10) *(for $r \neq 0$) and in* (11), *if and only if $\lambda(z) \equiv 1$; that is, if and only if S is a developable piece of surface and is a geodesic circle given in isometric representation.*

Theorem 3 is a special case of Theorem 4.

A generalization of Theorem 1 to isothermic maps on minimal surfaces, similar to the above generalization of Theorem 3 to isothermic maps on surfaces of non-positive curvature, previously has been given.* Since $K \leqq 0$ on minimal surfaces, Theorem 4 applies in particular to minimal surfaces.

An easily obtained generalization of Liouville's theorem is that if $F(z)=F(u+iv)$ is a harmonic function of u and v in the entire finite plane and is bounded, then $F(z)$ is identically constant. It follows that if the functions (8), not necessarily satisfying (9), are harmonic in the entire finite plane, and if the corresponding surface S is bounded, then S reduces to a point. In particular, since the coordinate functions of a minimal surface in isothermic representation necessarily are harmonic, *if the functions* (8) *give an isothermic representation of the entire finite plane on a minimal surface S, and if S is bounded, then S reduces to a point.*

From Theorem 4 we shall obtain the following similar generalization to space of Liouville's theorem:

THEOREM 5. *Let the functions* (8) *satisfy* (9) *in the entire finite plane, and let the Gaussian curvature K be less than or equal to zero wherever K is defined on the corresponding surface S. If $l(r, \theta; \lambda)$ is bounded,*

$$l(r, \theta; \lambda) \leqq M,$$

for all (r, θ), then S reduces to a point.

* E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces*, Transactions of this Society, vol. 35 (1933), pp. 648–661.

Finally, we shall point out certain forms which our results show to be positive definite and shall add a further algebraic proof that these forms are positive definite.

2. **Subharmonic functions and functions of class PL.** In this section we present the definitions of subharmonic functions and functions of class PL and list the results concerning these functions which we shall use in the sequel.*

2.1. Let $g(z)$ be defined in a domain D (connected open set), and assume $-\infty \leqq g(z) < +\infty$ in D. Suppose that $g(z)$ is not identically equal to $-\infty$ in D, that $g(z)$ is upper semi-continuous in D, and that, for every domain D' lying together with its boundary B' in D and for every function $h(z)$ continuous in $D'+B'$, harmonic in D', and satisfying $h(z) \geqq g(z)$ on B', we have $h(z) \geqq g(z)$ in D'. Then $g(z)$ is said to be *subharmonic* in D.

2.2. A function $p(z)$, defined in a domain D, is said to be *of class* PL in D provided that $p(z) \geqq 0$ in D and $\log p(z)$ is subharmonic. It is understood that $\log p(z) = -\infty$ at points where $p(z) = 0$.

2.3. If $p(z)$ is of class PL in D, then $p(z)$ is subharmonic in D.

2.4. If a real function $g(z)$ admits continuous second derivatives, then a necessary and sufficient condition that $g(z)$ be subharmonic is that its Laplacian be non-negative:

$$\Delta g = g_{uu} + g_{vv} \geqq 0.$$

2.5. A subharmonic function $g(z)$ cannot attain its least upper bound at any (interior) point of its domain of definition D, unless $g(z)$ is identically constant.

2.6. If $g(z)$ is subharmonic in a domain D, and if Γ is a smooth Jordan curve in D, then $g(z)$ is summable on Γ as a function of the arc length.

2.7. The sum of a finite number of functions, subharmonic (or of class PL) in a domain D, is again a function subharmonic (or of class PL) in D.

2.8. The product of a finite number of functions of class PL in a domain D is again a function of class PL in D.

2.9. If the function $g(z)$ is subharmonic (or of class PL) in a domain D, then the sequence

$$g(z; k) = \frac{k^2}{\pi} \iint_{\rho < 1/k} g(z + \rho e^{i\theta}) \rho d\rho d\theta, \quad k = 1, 2, \cdots,$$

* For the definitions and results of this section, and for references to their sources see T. Radó, *Subharmonic Functions*, Springer, Berlin, 1937, particularly chaps.1–3.

has the following properties. Let D' be a domain lying together with its boundary in D. Then for large k the function $g(z; k)$ is defined, subharmonic (or of class PL), and continuous in D'; and

$$g(z; k) \to g(z) \tag{12}$$

with

$$g(z; k) \geqq g(z; k+1) \tag{13}$$

in D'. We indicate that (12) and (13) both hold by writing $g(z; k) \searrow g(z)$.

2.10. Let there be given a domain D and a sequence of functions $\{g_n(z)\}$ such that if D' is any domain lying together with its boundary in D, then for large n the functions $g_n(z)$, $g_{n+1}(z)$, $\cdots$ are defined and subharmonic (or of class PL) in D'. If $\{g_n(z)\}$ converges uniformly in D' to a function $g(z)$, then $g(z)$ is subharmonic (or is either of class PL or identically zero) in D.

2.11. If, instead of converging uniformly in D', the sequence $\{g_n(z)\}$ of 2.10 satisfies

$$g_n(z) \geqq g_{n+1}(z)$$

in D', then $\{g_n(z)\}$ converges either to a subharmonic function or to $-\infty$ (or converges either to a function of class PL or to zero) throughout D.

3. **Lemma.** Of the three related results of this section, only 3.1, restricted to the easily proved special case in which $g(z)$ is continuous, will be used as a lemma in proving Theorem 2. The others are included for the sake of completeness.

3.1. *If $g(z)$ is subharmonic in $|z| < 1$, then the function*

$$h(re^{i\theta}) = \frac{1}{r}\int_0^r g(\rho e^{i\theta})d\rho, \qquad r \neq 0, \tag{14}$$

$$h(0) = g(0),$$

is again subharmonic in $r < 1$.

Proof. Suppose first that $g(z)$ is continuous. For any positive integer n, the function

$$h_n(re^{i\theta}) = \sum_{l=1}^{n} g(lre^{i\theta}/n)\frac{1}{n}$$

is subharmonic in $r < 1$ by 2.7. Further, the sequence $\{h_n(re^{i\theta})\}$ converges uniformly to $h(re^{i\theta})$ in any closed region in $r < 1$; hence, by 2.10 the function

$$h(re^{i\theta}) = \lim_{n\to\infty} h_n(re^{i\theta})$$

is subharmonic in $r<1$.

If $g(z)$ is a general subharmonic function, we can no longer consider $g(z)$ as a limit of Riemann sums, but may proceed as follows. The function $g(z; k)$, defined in 2.9, is defined, subharmonic, and continuous in $|z|<1-1/k$. Hence by the analysis of the preceding paragraph, the function

$$h(re^{i\theta}; k) = \frac{1}{r}\int_0^r g(\rho e^{i\theta}; k)d\rho$$

is subharmonic in $r<1-1/k$. Further, by 2.9,

$$g(\rho e^{i\theta}; k) \searrow g(\rho e^{i\theta}),$$

and by 2.6 the integral (14) exists; hence

$$\int_0^r g(\rho e^{i\theta}; k)d\rho \geqq \int_0^r g(\rho e^{i\theta})d\rho,$$

and*

$$\lim_{k\to\infty} \int_0^r g(\rho e^{i\theta}; k)d\rho = \int_0^r g(\rho e^{i\theta})d\rho;$$

that is,

$$h(re^{i\theta}; k) \searrow h(re^{i\theta}).$$

Therefore, by 2.11, $h(re^{i\theta})$ is subharmonic.

Similarly, we have the following result:

3.2. *If $p(z)$ is of class* PL *in $|z|<1$, then the function*

$$q(re^{i\theta}) = \frac{1}{r}\int_0^r p(\rho e^{i\theta})d\rho, \qquad r\neq 0,$$

$$q(0) = p(0),$$

is again of class PL *in $r<1$.*

3.3. *If $p(z)$ is of class* PL *in $|z|<1$, then the function*

$$t(re^{i\theta}) = \int_0^r p(\rho e^{i\theta})d\rho$$

is again of class PL *in $r<1$.*

* See for instance S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 63.

PROOF. Since

$$t(re^{i\theta}) = rq(re^{i\theta}),$$

our result follows from 2.8, 3.2, and the fact that the function $r = |z|$ is of class PL.

However, the hypothesis that $g(z)$ is subharmonic for $|z| < 1$ does not imply that the function

$$k(re^{i\theta}) = \int_0^r g(\rho e^{i\theta})\, d\rho$$

is subharmonic for $r < 1$. For example, the function

$$g(z) = R(z) + 1 = u + 1$$

is (positive and) subharmonic for $|z| < 1$. But for the corresponding function

$$k(re^{i\theta}) = \frac{r^2}{2}\cos\theta + r,$$

we have

$$\Delta k(re^{i\theta}) = \frac{3}{2}\cos\theta + \frac{1}{r};$$

in particular,

$$\Delta k\left(\frac{3}{4}e^{i\pi}\right) = -\frac{3}{2} + \frac{4}{3} = -\frac{1}{6} < 0,$$

so that, by 2.4, $k(re^{i\theta})$ is not subharmonic throughout $r < 1$.

4. **Proof of Theorem** 2. Under the hypotheses of Theorem 2, the function

$$h(re^{i\theta}) = \frac{1}{r}\, l(r, \theta; g), \qquad r \neq 0,$$

$$h(0) = g(0),$$

is subharmonic for $r < 1$ by 3.1. From (3) we have

$$\limsup_{\rho \to 1,\ \rho < 1} h(\rho e^{i\theta}) \leqq 1;$$

whence by 2.5

$$h(re^{i\theta}) \leqq 1, \tag{15}$$

the sign of equality holding either throughout the circle $r<1$ or nowhere in $r<1$. Now (4) and (5) follow from (15) and the definition of $h(re^{i\theta})$.

5. **On Theorems 3 and 4.** If $f(z)$ is analytic, then $|f(z)|$ is of class PL. Indeed, if $f(z)\neq 0$ in a domain D, then $\log|f(z)|$, which is the real part of the analytic function $\log f(z)$, is harmonic there, and hence, by 2.4, is subharmonic.

Again, if for the functions (8) the equations (9) are satisfied, then the Gaussian curvature K of the corresponding surface S is given, wherever it is defined, by

$$K = -\frac{1}{\lambda}\Delta \log \lambda.^*$$

Hence, by 2.4, $K\leqq 0$ wherever K is defined on S if and only if $\lambda(z)$ is of class PL in $|z|<1$.

Since $|f'(z)|$ and $\lambda(z)$ are of class PL, they are subharmonic by 2.3, so that Theorems 3 and 4 follow as particular cases of Theorem 2. Indeed, Theorem 3 is a particular case of Theorem 4, with $K\equiv 0$.

Though $|f'(z)|$ and $\lambda(z)$ are of class PL, we have used only the weaker condition that they are subharmonic. There actually exist conformal maps on surfaces S with $K>0$ for which $\lambda(z)$ is subharmonic, and for these maps Theorem 4 still holds. However, $\lambda(z)$ is subharmonic for all isothermic maps of z domains on a surface S if and only if $K\leqq 0$ wherever K is defined on S;† so the condition that $\lambda(z)$ is subharmonic can be used for the surface S irrespective of the isothermic parameters if and only if $K\leqq 0$ wherever K is defined on S.

6. **Proof of Theorem 5.** Let the functions (8) satisfy the hypotheses of Theorem 5, and let $Z=z/R$, where R is positive. For $|z|=r<R$, the functions

$$X_j(Z) = x_j(z)/M, \qquad j = 1, 2, 3,$$

as functions of Z, satisfy the conditions of Theorem 4. Hence, for $r<R$

$$l(r, \theta; \lambda) < M|Z| = Mr/R.$$

Keeping $z=re^{i\theta}$ constant and letting R tend to ∞, we obtain

$$l(r, \theta; \lambda) = 0.$$

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* See E. F. Beckenbach and T. Radó, *Subharmonic functions and surfaces of negative curvature*, Transactions of this Society, vol. 35 (1933), pp. 662–674.

† E. F. Beckenbach, *On subharmonic functions*, Duke Mathematical Journal, vol. 1 (1935), pp. 480–483.

7. **On positive definite forms.** If the map of $|z|<1$ given by $w=f(z)$ is everywhere conformal, so that $f'(z)\neq 0$, then we may write

$$f'(z) = [\phi(z)]^2,$$

where $\phi(z)$ is analytic in $|z|<1$. Let

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n;$$

then

$$h(r, \theta; |f'|) = \frac{1}{r} l(r, \theta; \phi\bar{\phi})$$

$$= \sum_{n,m=0}^{\infty} \frac{a_n \bar{a}_m}{n+m+1} r^{n+m} e^{(n-m)i\theta},$$

where $\bar{\alpha}$ denotes the complex number conjugate to α.

Direct computation yields

$$\Delta h(r, \theta; \phi\bar{\phi}) = \sum_{n,m=1}^{\infty} a_n \bar{a}_m r^{n+m-2} e^{(n-m)i\theta} \frac{4nm}{n+m+1}$$

$$= \sum_{n,m=1}^{\infty} \frac{b_n \bar{b}_m}{n+m+1}, \tag{16}$$

where $b_n = 2na_n r^{n-1} e^{ni\theta}$.

Since, by 3.1, $h(r, \theta; \phi\bar{\phi})$ is subharmonic, the expression (16) is non-negative by 2.4. On the other hand, we shall show directly that (16) is non-negative and thus obtain an alternative proof of Lemma 3.1 for the special case under consideration.

That the Hermitian form (16) is positive definite follows from the identity

$$\sum_{n,m=1}^{\infty} \frac{b_n \bar{b}_m}{n+m+1} = \int_0^1 \left| \sum_{n=1}^{\infty} b_n \rho^n \right|^2 d\rho$$

$$= \frac{4}{r^3} \int_0^r |\phi'(\rho e^{i\theta})|^2 \rho d\rho,$$

or equally well from the identity*

$$\sum_{n,m=1}^{\infty} \frac{b_n \bar{b}_m}{n+m+1} = \sum_{j=0}^{\infty} \left| \sum_{n=1}^{\infty} \frac{n^j}{(n+1)^{j+1}} b_n \right|^2.$$

* Suggested by Professor H. E. Bray.

The stronger result of 3.3 shows that $l(r, \theta; \phi\bar{\phi})$ is of class PL; whence

$$[l(r, \theta; \phi\bar{\phi})]^2 \Delta \log l(r, \theta; \phi\bar{\phi})$$

$$(17) \quad = \sum_{g,h,j,k=1}^{\infty} \frac{a_g \bar{a}_h a_j \bar{a}_k (g-h)^2 (j-k)^2}{(g+h+1)(j+k+1)(g+k+1)(h+j+1)} \cdot r^{g+h+j+k} e^{i(g-h+j-k)} \geqq 0.$$

On the other hand, it can be shown directly, by an extension of the above identities, that (17) is positive definite.

THE RICE INSTITUTE

SOME ITERATED INTEGRALS IN THE FRACTIONAL CALCULUS

M. S. MACPHAIL

1. **Introduction.** A considerable amount of attention has been devoted to integrals of fractional order, both in regard to their applications and to the conditions for their existence.* We shall denote the fractional integral of order α by

$$(1) \quad {}_T I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_T^t (t-v)^{\alpha-1} f(v) dv, \qquad \alpha > 0, t > T,$$

and it is the purpose of this paper to give some formulas which may be of use in manipulating these integrals. We shall prove that under certain conditions the following relations hold:

$$(2) \quad \int_T^{\infty} \frac{{}_T I_t^{\alpha} f(t)}{t^{k+\alpha}} dt = \frac{\Gamma(k)}{\Gamma(k+\alpha)} \int_T^{\infty} \frac{f(t)}{t^k} dt, \qquad \alpha > 0,$$

$$(3) \quad \int_T^{\infty} e^{-kt} {}_T I_t^{\alpha} f(t) dt = k^{-\alpha} \int_T^{\infty} e^{-kt} f(t) dt, \qquad \alpha > 0,$$

$$(4) \quad \int_T^{\infty} \cos kt \, {}_T I_t^{\alpha} f(t) dt = k^{-\alpha} \int_T^{\infty} \cos (kt + \pi\alpha/2) f(t) dt, \quad 0 < \alpha < 1,$$

and (4) holds when cosine is replaced by sine. As an application we

* A bibliography is given by H. T. Davis, *Application of fractional operators to functional equations*, American Journal of Mathematics, vol. 49 (1927), pp. 123–142. See also J. D. Tamarkin, *On integrable solutions of Abel's integral equation*, Annals of Mathematics, (2), vol. 31 (1930), pp. 219–229.

shall evaluate by a new method certain well known integrals involving Bessel functions. It will be noticed that when $\alpha=1$ the formulas (2) and (3) agree with the results of integrating by parts in the ordinary way; they may therefore be regarded as extensions to fractional orders of integration of the familiar formula for integration by parts. The vanishing of the integrated term leads to some general theorems which are stated as corollaries to Theorem 1 below.

2. **The proofs of the above formulas.** The first formula may be stated in a theorem as follows:

THEOREM 1. *Assume* $T\geqq 0, \alpha>0, k>0$. *Then*

$$\int_T^\infty \frac{{}_TI_t^\alpha f(t)}{t^{k+\alpha}}\,dt = \frac{\Gamma(k)}{\Gamma(k+\alpha)}\int_T^\infty \frac{f(t)}{t^k}\,dt, \tag{5}$$

provided either that the right-hand side exists in the Lebesgue sense, or that $f(t)$ *is continuous and the right-hand side convergent, not necessarily absolutely.*

PROOF. We have

$$\begin{aligned}\Gamma(\alpha)\int_T^\mu \frac{{}_TI_t^\alpha f(t)}{t^{k+\alpha}}\,dt &= \int_T^\mu \frac{dt}{t^{k+\alpha}}\int_T^t (t-v)^{\alpha-1}f(v)dv\\ &= \int_T^\mu f(v)dv\int_v^\mu \frac{(t-v)^{\alpha-1}}{t^{k+\alpha}}\,dt\ ^*\\ &= \int_T^\mu f(v)dv\int_v^\mu t^{-k-1}(1-v/t)^{\alpha-1}dt\\ &= \int_T^\mu \frac{f(v)}{v^k}\,dv\int_{v/\mu}^1 u^{k-1}(1-u)^{\alpha-1}du,\end{aligned}$$

on setting $v/t=u$. If the right-hand side of (5) is absolutely convergent, we may let μ become infinite, justifying the process by inverting the order of integration, taking the positive and negative parts of $f(v)$ separately, and using the theorem of monotonic convergence;† our theorem then follows on evaluation of the u integral. Otherwise a more elaborate process is necessary. We integrate by parts, setting

$$\int_T^v \frac{f(v)}{v^k}\,dv = F(v).$$

* The inversion is easily justified. See, for example, E. C. Titchmarsh, *The Theory of Functions*, p. 398, example 25.

† Titchmarsh, loc. cit., p. 346.

This gives

$$\left[F(v)\int_{v/\mu}^{1} u^{k-1}(1-u)^{\alpha-1}du\right]_T^{\mu} + \frac{1}{\mu}\int_T^{\mu} F(v)(v/\mu)^{k-1}(1-v/\mu)^{\alpha-1}dv.$$

The integrated term vanishes at both limits; and on the assumption that $F(v)$ approaches a limit, say V, as $v\to\infty$, we shall now show that the last integral approaches

$$V\frac{\Gamma(k)\Gamma(\alpha)}{\Gamma(k+\alpha)}$$

as $\mu\to\infty$. Write $F(v)=V+h(v)$, where $|h(v)|<\epsilon$ if $v>v_0$, and in any case $|h(v)|<A$ (A denoting a constant, not necessarily the same each time it is used). We divide the integral into four parts as follows. In the first place,

$$\frac{1}{\mu}\int_0^{\mu} V(v/\mu)^{k-1}(1-v/\mu)^{\alpha-1}dv = \frac{V}{\mu^{k+\alpha-1}}\int_0^{\mu} v^{k-1}(\mu-v)^{\alpha-1}dv$$

$$= V\frac{\Gamma(k)\Gamma(\alpha)}{\Gamma(k+\alpha)}.$$

Also, if $\mu>T$,

$$\frac{1}{\mu}\int_0^{T} V(v/\mu)^{k-1}(1-v/\mu)^{\alpha-1}dv = \frac{V}{\mu^k}\int_0^{T} v^{k-1}(1-v/\mu)^{\alpha-1}dv < \frac{A}{\mu^k}.$$

Supposing further that $\mu>v_0>T$, we write

$$\frac{1}{\mu}\int_T^{v_0} |h(v)|(v/\mu)^{k-1}(1-v/\mu)^{\alpha-1}dv < \frac{A}{\mu^k}\int_T^{v_0} v^{k-1}(1-v/\mu)^{\alpha-1}dv < \frac{A}{\mu^k},$$

and finally,

$$\frac{1}{\mu}\int_{v_0}^{\mu} |h(v)|(v/\mu)^{k-1}(1-v/\mu)^{\alpha-1}dv < \epsilon\frac{\Gamma(k)\Gamma(\alpha)}{\Gamma(k+\alpha)}.$$

The theorem now follows on combining the results of the four preceding equations.

COROLLARY 1. *If the integral*

$$\int_T^{\infty}\frac{f(v)}{v^k}dv$$

converges, then

$$\frac{1}{\mu^k}\int_T^\mu f(v)dv \to 0$$

as $\mu \to \infty$.

For, if $F(v)$ denotes an indefinite integral of $f(v)$, we have, on integrating by parts in the usual way,

$$\int_T^\mu \frac{f(v)}{v^k}dv = \left[\frac{F(v)}{v^k}\right]_T^\mu + k\int_T^\mu \frac{F(v)}{v^{k+1}}dv.$$

Comparing this with the result of the above theorem (with $\alpha = 1$), we see that the integrated term must vanish at $\mu = \infty$, which proves the corollary. That the converse proposition need not be true may be seen by taking $f(v) = (\log v)^{-1}$, $k = 1$.

This corollary leads us to the following more general result:

COROLLARY 2. *If the integral*

$$\int_T^\infty g(v)f(v)dv$$

converges, then

$$g(\mu)\int_T^\mu f(v)dv \to 0$$

as $\mu \to \infty$, *the function* $g(v)$ *being any function that tends monotonically to zero as* $v \to \infty$.

Suppose first that $g(v)$ is monotonically decreasing in the strict sense, and set $g(v) = u^{-1}$. Let the inverse relation be $v = G(u)$, and suppose further that $G(u)$ is the integral of a function $G'(u)$; then by the conditions of this corollary the integral

$$\int_{1/g(T)}^\infty u^{-1}f\{G(u)\}G'(u)du$$

exists.* Hence by Corollary 1 (with $k = 1$) we have

$$\mu^{-1}\int_{1/g(T)}^\mu f\{G(u)\}G'(u)du \to 0.$$

Changing the variable back to v, we have

$$\mu^{-1}\int_T^{G(\mu)} f(v)dv \to 0,$$

* Titchmarsh, loc. cit., p. 377.

or, what is the same thing,

$$g(\mu)\int_T^\mu f(v)dv \to 0,$$

as required. Next, suppose that $g(v)$ is a step function tending to zero. Evidently we may replace $g(v)$ by a steadily decreasing function (in the strict sense) $\bar{g}(v)$, such that

$$\left|\int_T^\infty g(v)f(v)dv - \int_T^\infty \bar{g}(v)f(v)dv\right| < \epsilon$$

for any value of $\epsilon>0$, and at the same time,

$$\lim_{v\to\infty} \bar{g}(v)/g(v) = 1.$$

The corollary will then be true for $\bar{g}(v)$ and therefore for $g(v)$. Finally, let $g(v)$ be any function tending monotonically to zero, not necessarily in the strict sense. Then we can replace $g(v)$ by a step function and this in turn by a steadily decreasing function in the strict sense. The corollary now follows as before.

COROLLARY 3.* *Let $t_1, t_2, \cdots$ be a sequence tending monotonically to zero, and let $u_1, u_2, \cdots$ be any sequence such that $\sum_{n=1}^\infty t_n u_n$ exists. Then*

$$t_m\sum_{n=1}^m u_n \to 0$$

as $m\to\infty$.

For each sum may be represented as the integral of a step function. The result then follows from Corollary 2.

THEOREM 2. *Assume $\alpha>0$, $k>0$. Then*

$$\int_T^\infty e^{-kt}{}_T I_t^\alpha f(t)dt = k^{-\alpha}\int_T^\infty e^{-kt}f(t)dt,$$

provided either that the right-hand side exists in the Lebesgue sense, or that $f(t)$ is continuous and the right-hand side convergent, not necessarily absolutely.

PROOF. We have the following relations:

* A particular case of this corollary, with $t_n=n^{-k}$, is given by Pólya and Szegö, *Aufgaben und Lehrsätze*, vol. 1, part 1, exercise 75.

$$\Gamma(\alpha)\int_T^\mu e^{-kt}{}_TI_t^\alpha f(t)dt = \int_T^\mu e^{-kt}dt\int_T^t (t-v)^{\alpha-1}f(v)dv$$

$$= \int_T^\mu f(v)dv\int_v^\mu e^{-kt}(t-v)^{\alpha-1}dt$$

$$= \int_T^\mu f(v)dv\int_0^{\mu-v} e^{-k(u+v)}u^{\alpha-1}du$$

$$= \int_T^\mu e^{-kv}f(v)dv\int_0^{\mu-v} e^{-ku}u^{\alpha-1}du$$

$$\to \Gamma(\alpha)k^{-\alpha}\int_T^\infty e^{-kv}f(v)dv,$$

as $\mu\to\infty$, the process being justified as in Theorem 1. This proves the theorem.

THEOREM 3. *Assume* $k>0$, $0<\alpha<1$. *Then*

$$\int_T^\infty \cos kt\,{}_TI_t^\alpha f(t)dt = k^{-\alpha}\int_T^\infty \cos(kt+\pi\alpha/2)f(t)dt,$$

provided that either the right-hand side exists in the Lebesgue sense or $f(t)$ *is continuous and the right-hand side convergent, not necessarily absolutely. The same is true when the cosine is replaced by sine throughout.*

PROOF. We have

$$\Gamma(\alpha)\int_T^\mu \cos kt\,{}_TI_t^\alpha f(t)dt = \int_T^\mu \cos kt\,dt\int_T^t (t-v)^{\alpha-1}f(v)dv$$

$$= \int_T^\mu f(v)dv\int_v^\mu (t-v)^{\alpha-1}\cos kt\,dt$$

$$= k^{-\alpha}\int_T^\mu f(v)dv\int_0^{k(\mu-v)} u^{\alpha-1}\cos(u+kv)du$$

$$= k^{-\alpha}\left\{\int_T^\mu f(v)\cos kv\,dv\int_0^{k(\mu-v)} u^{\alpha-1}\cos u\,du\right.$$

$$\left. -\int_T^\mu f(v)\sin kv\,dv\int_0^{k(\mu-v)} u^{\alpha-1}\sin u\,du\right\}$$

$$\to \Gamma(\alpha)k^{-\alpha}\int_T^\infty \cos(kv+\pi\alpha/2)f(v)dv,$$

as $\mu \to \infty$, the process being justified as in Theorem 1. The corresponding theorem, with cosine replaced by sine, may be proved in a similar way.

It will be noticed that the above proof breaks down when $\alpha = 1$. If indeed the theorem were true for $\alpha = 1$, we should have as a corollary that

$$\cos k\mu \int_T^\mu f(t)dt \to 0,$$

which is clearly not true in general.

3. **Applications of the theorems.** We now apply these theorems to the evaluation of the integrals*

$$(6) \qquad \int_0^\infty x^{-n-m} J_n(ax)dx, \qquad n > -1/2,\ 0 < m < 1,$$

$$(7) \qquad \int_0^\infty e^{-kx^2} x^{n+1} J_n(ax)dx, \qquad n > -1/2,\ k > 0,$$

$$(8) \qquad \int_0^\infty x^{n+1} \cos kx^2 J_n(ax)dx, \qquad -1/2 < n < 1/2,\ k > 0,$$

$$\int_0^\infty x^{n+1} \sin kx^2 J_n(ax)dx, \qquad -1/2 < n < 1/2,\ k > 0.$$

It is found that Bessel functions lend themselves readily to the methods of the fractional calculus. W. O. Pennell† has shown that

$$(9) \qquad J_n(at^{1/2}) = (a/2)^{n-1}\pi^{-1/2} t^{-n/2} {}_0I_t^{n-1/2} \sin at^{1/2}.$$

This was proved by expanding $\sin at^{1/2}$ in a power series and integrating fractionally term by term. It holds for all values of n, positive and negative, provided we write as usual $I^{-\alpha} = D^p I^{p-\alpha}$, p being an integer such that $p > \alpha \geqq p-1 \geqq 0$, and D^p denoting repeated differentiation in the usual sense. If $n > -1/2$, we may also derive (9) by taking Poisson's integral

$$J_n(at) = \frac{2(at/2)^n}{\Gamma(n + 1/2)\Gamma(1/2)} \int_0^1 (1 - u^2)^{n-1/2} \cos atu\, du,$$

and setting $tu = v^{1/2}$. This gives

* An evaluation of integrals (6) and (7) is to be found in Watson, *Bessel Functions*, chap. 13, pp. 393–394; or see Gray and Mathews, *Bessel Functions*, 1st edition, formulas (163) and (162), or 2d edition, formula (8) on page 66 and formula (19) on page 71. In each of these books references to the original papers of Weber and others are given.

† This Bulletin, vol. 38 (1932), pp. 115–122.

$$J_n(at) = \frac{(at/2)^n}{t^{2n}\Gamma(n+1/2)\Gamma(1/2)} \int_0^{t^2} (t^2 - v)^{n-1/2} \frac{\cos av^{1/2}}{v^{1/2}}\, dv;$$

whence, by changing t into $t^{1/2}$, we obtain

$$J_n(at^{1/2}) = (a/2)^n \pi^{-1/2} t^{-n/2} {}_0I_t^{n+1/2}(t^{-1/2} \cos at^{1/2}), \tag{10}$$

a formula equivalent to (9). It is now easy to evaluate the integrals under consideration. In the first place, we have

$$\int_0^\infty \frac{J_n(ax)}{x^{n+m}}\, dx = \frac{1}{2} \int_0^\infty \frac{J_n(at^{1/2})}{t^{n/2+m/2+1/2}}\, dt$$

$$= \frac{1}{2}(a/2)^n \pi^{-1/2} \int_0^\infty \frac{1}{t^{n+m/2+1/2}} {}_0I_t^{n+1/2} \frac{\cos at^{1/2}}{t^{1/2}}\, dt$$

by (10). Applying Theorem 1, we find that this is equal to

$$\frac{1}{2}(a/2)^n \pi^{-1/2} \frac{\Gamma(m/2)}{\Gamma(n+m/2+1/2)} \int_0^\infty \frac{\cos at^{1/2}}{t^{m/2+1/2}}\, dt$$

$$= (a/2)^n \pi^{-1/2} \frac{\Gamma(m/2)}{\Gamma(n+m/2+1/2)} \int_0^\infty \frac{\cos ax}{x^m}\, dx$$

$$= (a/2)^n \pi^{-1/2} \frac{\Gamma(m/2)}{\Gamma(n+m/2+1/2)} a^{m-1}\Gamma(1-m) \sin \pi m/2.$$

Simplifying by routine methods, we obtain finally

$$\int_0^\infty \frac{J_n(ax)}{x^{n+m}}\, dx = a^{n+m-1} 2^{-n-m} \frac{\Gamma(1/2 - m/2)}{\Gamma(n+m/2+1/2)}.$$

The same method may be applied to the integral (7). We have

$$\int_0^\infty e^{-kx^2} x^{n+1} J_n(ax)\, dx = \frac{1}{2} \int_0^\infty e^{-kt} t^{n/2} J_n(at^{1/2})\, dt$$

$$= \frac{1}{2}(a/2)^n \pi^{-1/2} \int_0^\infty e^{-kt} {}_0I_t^{n+1/2}(t^{-1/2} \cos at^{1/2})\, dt,$$

by (10). Applying Theorem 2 we find that this is equal to

$$\frac{1}{2}(a/2)^n \pi^{-1/2} k^{-(n+1/2)} \int_0^\infty e^{-kt} t^{-1/2} \cos at^{1/2} dt$$

$$= (a/2)^n \pi^{-1/2} k^{-(n+1/2)} \int_0^\infty e^{-kx^2} \cos ax\, dx$$

$$= (a/2)^n \pi^{-1/2} k^{-(n+1/2)} e^{-a^2/4k} \int_0^\infty e^{-kx^2} dx,$$

as is readily proved by a contour integration method.* Hence, finally,

$$\int_0^\infty e^{-kx^2} x^{n+1} J_n(ax) dx = (2k)^{-n-1} a^n e^{-a^2/4k}.$$

Similarly, by using Theorem 3, we find that

$$\int_0^\infty x^{n+1} \cos kx^2 J_n(ax) dx$$
$$= 2^{1/2}(2k)^{-n-1} a^n \{\cos (\pi n/2 + \pi/4 - a^2/4k) - \sin (\pi n/2 + \pi/4 - a^2/4k)\}$$

and that

$$\int_0^\infty x^{n+1} \sin kx^2 J_n(ax) dx$$
$$= 2^{1/2}(2k)^{-n-1} a^n \{\cos (\pi n/2 + \pi/4 - a^2/4k) + \sin (\pi n/2 + \pi/4 - a^2/4k)\},$$

provided $-1/2 < n < 1/2$. These examples will serve to indicate some of the uses of the theorems given above.

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* Whittaker and Watson, *Modern Analysis*, 4th edition, p. 114, example 3.

TRANSFORMATION OF BASES FOR RELATIVE LINEAR SETS*

MARGARETE C. WOLF†

The definitions of linear independence, dependence, and extension of sets of vectors relative to a matrix which are used in this paper were recently introduced by M. H. Ingraham,‡ and are not repeated here. The purpose of the present paper is to study the structure of basal elements of a linear extension relative to a matrix. There are developed necessary and sufficient conditions which the elements of a matrix of a transformation must satisfy so that one set of basal elements of a vector space can be transformed into another basal set.

It is assumed throughout this paper that the elements of the matrices and the vectors, as well as the coefficients of the polynomials, are in a field $\mathfrak{F}$. The following theorem is used but stated without proof:§

THEOREM 1. *If $\xi_1, \xi_2, \cdots, \xi_k$ and $\eta_1, \eta_2, \cdots, \eta_l$ are two sets of vectors ($n \times 1$ matrices), each of which is linearly independent relative to an $n \times n$ matrix M, such that $L_M(\xi_1, \xi_2, \cdots, \xi_k) = L_M(\eta_1, \eta_2, \cdots, \eta_l)$, if h_{1i} and h_{2i} are, respectively, the minimum polynomials associated with ξ_i and η_i relative to M, and if g is an irreducible polynomial and t any positive integer, then the number of polynomials h_{1i} divisible by g^t is equal to the number of h_{2i} divisible by g^t.*

This is essentially the only restriction on the polynomials h_{1i} and h_{2i}.

THEOREM 2. *If $\xi_1, \xi_2, \cdots, \xi_k$ is a proper base relative to a matrix M for the space $L_M(\xi_1, \xi_2, \cdots, \xi_k)$, if the minimum polynomial associated with ξ_i relative to M is g^{t_i}, a power of an irreducible polynomial g, and if $\eta_i = \sum_{j=1}^{k} f_{ij}(M)\xi_j$, $(i = 1, 2, \cdots, k)$, where the f_{ij} are polynomials with coefficients in the field $\mathfrak{F}$, necessary and sufficient conditions that the η_i form a proper base relative to M are that the set of polynomials minimally associated with the η_i is exactly the set g^{t_i} in some order, and*

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† The work of M. C. Wolf was supported by a grant from the Wisconsin Alumni Research Foundation. The topic was suggested by M. H. Ingraham, who also contributed valuable ideas and criticisms when the paper was presented to a seminar.

‡ M. H. Ingraham and M. C. Wolf, *Relative linear sets and similarity of matrices whose elements belong to a division algebra*, Transactions of this Society, vol. 42 (1937), pp. 16–31.

§ M. H. Ingraham and M. C. Wolf, loc. cit.

that the $k \times k$ matrix (f_{ij}) is of rank k in the ring of polynomials reduced modulo g.

The necessity of the conditions will be proved first. Obviously the minimum polynomial associated with η_i is a power of g; call it g^{t_i}. From Theorem 1 it follows that the set of integers t_i is equal to the set s_i. Let η and ξ be vectors of vectors with η_i and ξ_i, respectively, as the ith element. Let $F=(f_{ij})$ be the matrix with the polynomial f_{ij} in the ith row and jth column, and $F(M)$ the matrix of matrices $(f_{ij}(M))$. The relations $\eta_i=\sum_{j=1}^{k} f_{ij}(M)\xi_j$, $(i=1, 2, \cdots, k)$, can be written as $\eta=F(M)\xi$. A necessary condition that $L_M(\xi_1, \xi_2, \cdots, \xi_k) = L_M(\eta_1, \eta_2, \cdots, \eta_k)$ is that a matrix G exist such that $\xi=G(M)\eta$. Hence it is necessary that G exist such that $\xi=G(M)F(M)\xi$. The matrix GF is square with elements w_{ij}, polynomials with coefficients in $\mathfrak{F}$, and $\xi_i=\sum_{j=1}^{k} w_{ij}(M)\xi_j$. The vectors $\xi_1, \xi_2, \cdots, \xi_k$ are by hypothesis linearly independent relative to M; hence $w_{jj}\equiv 1 \pmod{g^{t_j}}$, and $w_{ij}\equiv 0 \pmod{g^{t_j}}$, $(i\neq j)$. Hence it is necessary that $GF\equiv I \pmod{g}$ and that the determinant $|GF| \equiv 1 \pmod{g}$. The matrices F and G are square; hence $|GF| = |G|\,|F| \equiv 1 \pmod{g}$. Consequently $|F| = |(f_{ij})| \not\equiv 0 \pmod{g}$.

The conditions are sufficient, for if $|F| \not\equiv 0 \pmod{g}$, then for any positive integer t, $(|F|, g^t)=1$, and there exists a polynomial denoted by $|F|^{-1}$, such that $|F| \cdot |F|^{-1} \equiv 1 \pmod{g^t}$. Let $G=|F|^{-1}$ adj F, where adj F is the adjoint of the matrix F. Then $GF\equiv I \pmod{g^t}$. Hence $G(M)F(M)\xi=\xi$; that is, $G(M)\eta=\xi$, and consequently $\eta_1, \eta_2, \cdots, \eta_k$ form a base relative to M for $L_M(\xi_1, \xi_2, \cdots, \xi_k)$. Let g^{s_i} be minimally associated relative to M with η_i, where the set of s_i, $(i=1, 2, \cdots, k)$, is equal to the set of t_i, $(i=1, 2, \cdots, k)$, in some order. The proof that these η_i are linearly independent relative to M is based on the theorem that the number of basal elements in a proper base for a space relative to a field $\mathfrak{F}$ is independent of the base.

From Theorem 1 and from the fact that it is necessary that $|(f_{ii})| \not\equiv 0 \pmod{g}$, the η_i may be rearranged so that the minimum polynomial associated with η_i is g^{t_i}, where $t_1 \geqq t_2 \geqq \cdots \geqq t_k$. Then η_i is of the form

$$
\begin{aligned}
\eta_i = {} & g^{t_1-t_i}(M)h_{i1}(M)\xi_1 + g^{t_2-t_i}(M)h_{i2}(M)\xi_2 + \cdots \\
& + g^{t_{i-1}-t_i}(M)h_{ii-1}(M)\xi_{i-1} \\
& + f_{ii}(M)\xi_i + f_{ii+1}(M)\xi_{i+1} + \cdots + f_{ik}(M)\xi_k,
\end{aligned} \tag{1}
$$

where $(f_{ii}, g)=1$. If $t_i > t_{i+1}$, the h_{ij} and the f_{ij} are arbitrary polynomials. However if $t_{i-s-1} > t_{i-s} = t_{i-s+1} = \cdots = t_i > t_{i+1}$, then the prin-

cipal minor determinant with diagonal elements $f_{i-s\,i-s}, \cdots, f_{i-1\,i-1}, f_{ii}$ cannot be congruent to zero modulo g. Obviously, if the vectors η_i are of the form (1), and if the above conditions on the h_{ij} and f_{ij} are satisfied, the sufficient conditions of Theorem 2 are fulfilled.

THEOREM 3. *If the set of vectors $\xi_1, \xi_2, \cdots, \xi_r$ is a proper base relative to a matrix M for $L_M(\xi_1, \xi_2, \cdots, \xi_r)$, if $\eta_k = \sum_{j=1}^{r} f_{kj}(M)\xi_j$, where the f_{kj} are polynomials with coefficients in $\mathfrak{F}$, and if $\prod_{i=1}^{u} g_i^{t_{ji}}$ and $\prod_{i=1}^{v} g_i^{s_{ki}}$ are minimally associated with ξ_j and η_k, respectively, where the g_i are polynomials irreducible in $\mathfrak{F}$, necessary and sufficient conditions that the η_k form a proper base for $L_M(\xi_1, \xi_2, \cdots, \xi_r)$ are*: (i) *for every i the set of non-zero integers s_{ki} is equal to the set of non-zero t_{ji} in some order, and* (ii) *for every i and for those v_i values of j for which g_i is a factor of the minimum polynomial associated with ξ_j, the v_i columns of (f_{kj}) are of rank v_i in the ring of polynomials reduced modulo g_i.*

Let exactly v_i of the polynomials minimally associated with the ξ_j have a factor g_i. Then as a consequence of Theorem 1 exactly v_i of the polynomials minimally associated with the η_k have a factor g_i. Let $p_i = \prod_{k \neq i} g_k^{c_k}$, where for every value of k the c_k is the maximum value of t_{jk}, $(j = 1, 2, \cdots, r)$.

After a rearrangement of the ξ_j and η_k it may be assumed that $p_i(M)\xi_j = p_i(M)\eta_j = 0$, $(j \geqq v_i + 1)$, and that $p_i(M)\xi_j \neq 0$, $p_i(M)\eta_j \neq 0$, $(j < v_i)$. If polynomials exist such that $\xi_j = \sum_k q_{jk}(M)\eta_k$, then

$$p_i(M)\xi_j = \sum_{k=1}^{v_i} q_{jk}(M)p_i(M)\eta_k.$$

The non-zero $p_i(M)\xi_j$ are linearly independent relative to M for every i. Therefore by Theorem 2 for $k, j = 1, 2, \cdots, v_i$, (f_{kj}) is of rank v_i in the ring of polynomials reduced modulo g_i. If $j \geqq v_i + 1$, then $f_{jk} \equiv 0 \pmod{g_i^{t_{ki}}}$ for $k = 1, 2, \cdots, v_i$, since $(p_i, g_i) = 1$ and since $p_i(M)\eta_j = 0$ when $j \geqq v_i + 1$. For different values of i those v_i columns, which are of rank v_i in the ring of polynomials reduced modulo g_i, may overlap.

That the conditions of the theorem are sufficient for a proper base follows in a similar manner, since a greatest common divisor of the p_i is 1.

The problem of the transformation of bases of relative linear sets is being studied also in the case where $\mathfrak{F}$ is a division algebra, not necessarily commutative.

THE UNIVERSITY OF WISCONSIN

A NOTE ON THE ELEMENTARY DIVISOR THEORY IN NON-COMMUTATIVE DOMAINS*

TADASI NAKAYAMA

Two main features of the classical elementary divisor theory are the reduction of a matrix with integral coefficients to a normal form, which is a diagonal form with certain properties, and the uniqueness of such a normal form. The former of these two was extended to non-commutative domains by B. L. van der Waerden† and J. H. M. Wedderburn,‡ and a further contribution in this line was made by N. Jacobson.§ Moreover, O. Teichmüller‖ showed recently that the so-called euclidean division process is unnecessary for the purpose and the weaker assumption that the domain is a principal ideal domain is sufficient. As for the second problem, namely the uniqueness problem, as it seems to me, little has been done in the non-commutative case except to show that the directly indecomposable components of the diagonal elements as a whole are, in virtue of the Krull-Remak-Schmidt theorem, unique up to similarity. In the present short note¶ we shall, generalizing a result in a joint note of K. Asano and the author,** see that the diagonal elements of a Jacobson-Teichmüller normal form themselves are determined uniquely up to similarity, although this uniqueness theorem is not so satisfactory and is essentially not so far from the uniqueness of the indecomposable components.

Let I be a (not necessarily commutative) domain of integrity†† in

* Presented to the Society, February 26, 1938. The writer is grateful to the referee whose comments enabled him to make some necessary changes.

† B. L. van der Waerden, *Moderne Algebra*, vol. 2, 1st edition, 1932.

‡ J. H. M. Wedderburn, *Non-commutative domains of integrity*, Journal für die reine und angewandte Mathematik, vol. 167 (1932).

§ N. Jacobson, *Pseudolinear transformations*, Annals of Mathematics, vol. 38 (1937).

‖ O. Teichmüller, *Der Elementarteilersatz für nichtkommutative Ringe*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1937.

¶ The abstract of the note has appeared in this Bulletin, abstract 44-3-118, but the present note consists of the content of the second half of the original one; the first half was omitted since the author found later that that part had been proved in Teichmüller, loc. cit. The note came out originally from an oral discussion with K. Asano and M. Takahasi.

** K. Asano and T. Nakayama, *Über halblineare Transformationen*, Mathematische Annalen, vol. 115 (1937).

†† That is, an associative ring with unit element 1 and without divisors of zero.

which all left and right ideals are principal; the left and right principal ideals generated by an element a we denote by $(a)_l$ and $(a)_r$, respectively. We have then the maximum condition for both left and right ideals, and we have also the so-called weakened minimum condition for both left and right ideals, that is, every descending chain of left or right ideals containing a fixed non-zero element is finite. Every element not a unit or the zero can be expressed as a product of a finite number of irreducible elements, and the number of the factors in such an expression is, in virtue of the Jordan-Hölder theorem, uniquely determined by the element.* We call this number the length of the element, as well as that of the left and right ideals generated by the element. The length of a unit we define to be 0. Furthermore, if a and b are both different from 0, then $(a)_r \cap (b)_r \neq 0$ and $(a)_l \cap (b)_l \neq 0$ (existence of the least common multiples); for the proof see Teichmüller, loc. cit. (Refer also to Asano and Nakayama, loc. cit., p. 89.)

The following elementary divisor theorem for I was proved in Teichmüller, loc. cit.: *Let M be a (m, n)-matrix with elements from I. Then there exist invertible square matrices U and V (of order m and n, respectively) such that $M_0 = UMV$ has the form*

$$M_0 = \begin{pmatrix} a_1 & & & & 0 \\ & a_2 & & & \\ & & \ddots & & \\ & & & a_t & \\ 0 & & & & 0 \end{pmatrix}, \qquad a_i \neq 0,$$

where $Ia_{i+1}I \subseteqq (a_i)_l \cap (a_i)_r$ $(= Ia_i \cap a_iI)$ for all $i = 1, 2, \cdots, t-1$. (This M_0 we shall call a *Jacobson-Teichmüller normal form* for M.)

Let us call, after Jacobson, an element a *bounded* if there is a two-sided ideal divisible† by a and different from 0. Furthermore, we call the (set-theoretically) greatest two-sided ideal divisible by a the *bound* of a in case a is bounded. Then the above condition for a_i's, which Teichmüller expressed by saying that a_{i+1} is totally divisible by a_i, can be expressed also in the following manner: All a_i except a_t are bounded, and the bound of a_i divides a_{i+1}, $(i = 1, 2, \cdots, t-1)$.

Two right (left) ideals $\mathfrak{r}_1$ and $\mathfrak{r}_2$ ($\mathfrak{l}_1$ and $\mathfrak{l}_2$) are called similar if the I-right (left)-moduli $I/\mathfrak{r}_1$ and $I/\mathfrak{r}_2$ ($I/\mathfrak{l}_1$ and $I/\mathfrak{l}_2$) are isomorphic. Two

* Not only the number of the factors but the factors themselves are determined uniquely up to similarity.

† We say that a two-sided ideal $\mathfrak{t}$ is divisible by a if $\mathfrak{t} \subseteqq (a)_l \cap (a)_r$. This will be the case as soon as either $\mathfrak{t} \subseteqq (a)_l$ or $\mathfrak{t} \subseteqq (a)_r$, as one can see easily.

elements a and b are said to be r-similar* (l-similar) if $(a)_r$ and $(b)_r$ ($(a)_l$ and $(b)_l$) are similar. But then r-similarity and l-similarity are coexistent,† and we can therefore simply speak of similarity of two elements.

THEOREM. *The number t of the non-zero diagonal elements $a_1, a_2, \cdots, a_t$ in a Jacobson-Teichmüller normal form is uniquely determined by the matrix M, and furthermore the elements $a_1, a_2, \cdots, a_t$ are determined by M uniquely up to similarity.*

Before we go to the proof of the theorem, let us notice that in so far as we deal with bounded elements only, the situation is quite similar to the usual formal arithmetic in a maximal order of a rational algebra. For example, two-sided ideals (not zero) form a commutative multiplicative system; if $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two two-sided ideals (not zero), then $\mathfrak{t}_1 \subseteq \mathfrak{t}_2$ if and only if there is a (uniquely determined) third two-sided ideal $\mathfrak{t}_3$ such that $\mathfrak{t}_1 = \mathfrak{t}_2\mathfrak{t}_3$ $(= \mathfrak{t}_3\mathfrak{t}_2)$; every two-sided ideal (not zero) may be expressed in a unique way as a product, as well as a direct cross cut, of powers of different (two-sided) prime ideals; a prime ideal is a direct cross cut of mutually similar irreducible right (left) ideals, and the number of such components, which is obviously independent of the special choice of the decomposition, is called the *capacity* of the prime ideal; if $\mathfrak{p}$ is a prime ideal with the capacity κ, then $\mathfrak{p}^\alpha$ is a direct cross cut of κ mutually similar (directly) indecomposable right (left) ideals of the length α; if $\mathfrak{p}^\alpha = \mathfrak{r}_1 \cap \mathfrak{r}_2 \cap \cdots \cap \mathfrak{r}_\kappa$ is a decomposition of $\mathfrak{p}^\alpha$ into indecomposable right ideals in the above sense, then $I, \mathfrak{r}_i \cup \mathfrak{p}, \mathfrak{r}_i \cup \mathfrak{p}^2, \cdots, \mathfrak{r}_i \cup \mathfrak{p}^{\alpha-1}, \mathfrak{r}_i$ exhaust all the right ideal divisors of $\mathfrak{r}_i$; every indecomposable right ideal generated by a bounded element divides a suitable power of a suitable prime ideal, say $\mathfrak{p}$, and if it has the length α, then it divides $\mathfrak{p}^\alpha$ but not $\mathfrak{p}^{\alpha-1}$, and moreover it is similar to the above $\mathfrak{r}_i$; every right ideal dividing a suitable power of a prime ideal of capacity κ is a direct cross cut of κ indecomposable right ideals ($I = (1)_r$ being allowed as components); every right ideal generated by a bounded element can be expressed uniquely as a direct cross cut of right ideals which divide the powers of different prime ideals. (The individual component in such decomposition is called the component of the given right ideal with respect to the corresponding prime ideal.)

* In Ore-Jacobson's sense r-similarity corresponds to left-similarity. The element a is r-similar to b if and only if there is an element c with $(ca)_r = (b)_r \cap (c)_r$, $(b)_r \cup (c)_r = I$.

† For this fact, although in our case it can be seen easily, see H. Fitting, *Über den Zusammenhang zwischen dem Begriff der Gleichartigkeit zweier Ideale und dem Äquivalenzbegriff der Elementarteilertheorie*, Mathematische Annalen, vol. 112 (1936).

All these statements can be verified as in the arithmetic in algebras,* and almost all of them are proved essentially in Jacobson, loc. cit. (Jacobson's case is a special case of ours, but the essential points are almost the same.) Simply expressed, the residue class ring of I with respect to a two-sided ideal (not zero) is one-rowed (einreihig) in the sense of G. Koethe.†

PROOF OF THE THEOREM. Let us consider as usual the module $\mathfrak{N}$ of all linear forms $x_1c_1+x_2c_2+\cdots+x_mc_m$ of m independent indeterminates x_i with coefficients from I, and define $\mathfrak{M}$ as the submodule of $\mathfrak{N}$ generated by n elements y_i with $(y_1, \cdots, y_n)=(x_1, \cdots, x_m)M$. Then t is the maximal number of linearly independent elements in $\mathfrak{M}$. Let $\mathfrak{N}_0$ be the submodule of $\mathfrak{N}$ consisting of all elements u such that $uc \in \mathfrak{M}$ for a suitable non-zero element c in I. Then $\mathfrak{N}_0/\mathfrak{M}$ is a direct sum of t cyclic submoduli with generators whose annihilators are $(a_1)_r$, $(a_2)_r, \cdots, (a_t)_r$, respectively. Now, let $\mathfrak{a}$ be the intersection of all principal right ideals generated by bounded left factors of a_t. Let us first prove that $(a_1)_r, \cdots, (a_{t-1})_r, \mathfrak{a}$ are determined by M uniquely up to similarity.‡ For that purpose, take a non-zero two-sided ideal $\mathfrak{t}$ divisible by $\mathfrak{a}$, and hence also by $(a_1)_r, \cdots, (a_{t-1})_r$. It is obvious that $(a_t)_r \cup \mathfrak{t}=\mathfrak{a}$. Consider then $\mathfrak{N}_0\mathfrak{t}$ and $\mathfrak{N}_0\mathfrak{t} \cup \mathfrak{M}$. The module $\mathfrak{N}_0/\mathfrak{N}_0\mathfrak{t} \cup \mathfrak{M}$ is a direct sum of cyclic submoduli with generators whose annihilators are $(a_1)_r, \cdots, (a_{t-1})_r, \mathfrak{a}$, respectively. Take any prime ideal $\mathfrak{p}$, and let κ be its capacity. Let us consider the $\mathfrak{p}$-components of $(a_1)_r, \cdots, (a_{t-1})_r, \mathfrak{a}$. Let the $\mathfrak{p}$-component of $(a_1)_r$ be a direct cross cut of κ indecomposable right ideals of the lengths (greater than or equal to zero) $l_{11} \leqq l_{12} \leqq \cdots \leqq l_{1\kappa}$, and let the $\mathfrak{p}$-component of $(a_2)_r$ be a direct cross cut of κ indecomposable right ideals of the lengths $l_{21} \leqq l_{22} \leqq \cdots \leqq l_{2\kappa}$, and so on. From our condition on $a_1, a_2, \cdots, a_t$ it is easy to see that then $l_{11} \leqq \cdots \leqq l_{1\kappa} \leqq l_{21} \leqq \cdots \leqq l_{2\kappa} \leqq l_{31} \leqq \cdots$. Now, if we have a second Jacobson-Teichmüller normal form of M with the diagonal elements $a_1', a_2', \cdots, a_t'$, then we can take $\mathfrak{t}$ so that $\mathfrak{t}$ is divisible also by all bounded left factors of a_t'. Then the Krull-Remak-Schmidt theorem concerning direct decompositions of a group (with chain conditions) applied to $\mathfrak{N}_0/\mathfrak{N}_0\mathfrak{t} \cup \mathfrak{M}$ shows that $l_{11}, \cdots, l_{1\kappa}, l_{21}, \cdots, l_{t\kappa}$ are, including their order, independent of the special choice of the normal form. Since $\mathfrak{p}$ is any prime ideal,

* See, for example, M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, 1935; cf. particularly chap. 6, §3.

† G. Koethe, *Verallgemeinerte Abelsche Gruppen mit hyperkomplexem Operatorenring*, Mathematische Zeitschrift, vol. 39 (1934).

‡ For the following cf. Asano and Nakayama, loc. cit., Theorem 9.

$(a_1)_r, \cdots, (a_{t-1})_r$ (and $\mathfrak{a}$) are determined uniquely up to similarity. Then the Krull-Remak-Schmidt theorem, applied now to $\mathfrak{N}_0/\mathfrak{M}$, shows that $(a_t)_r$ is also determined uniquely up to similarity.

REMARK. The above uniqueness theorem is unsatisfactory, since two diagonal forms (of the same type) with diagonal elements $a_1, a_2, \cdots$ and $a_1', a_2', \cdots$ are in general not equivalent (associate) even if $a_1, a_2, \cdots$ and $a_1', a_2', \cdots$ are similar in pairs. But if, moreover, a_1 (therefore also a_1') is a unit, then they are equivalent.*

THE INSTITUTE FOR ADVANCED STUDY

ON A MIXED BOUNDARY CONDITION FOR HARMONIC FUNCTIONS

HILLEL PORITSKY

In two recent notes in this Bulletin† (referred to below as I, II) I considered the boundary conditions

$$\frac{\partial u}{\partial n} + au = 0, \qquad a = \text{const.}, \tag{1}$$

for harmonic functions, investigating in particular the "reflection" of singularities across a plane at which (1) obtains and indicating several applications of the results.

Dr. A. Weinstein has kindly called my attention to an application of (1) that I have overlooked, namely, to the problem of gravity surface waves of liquids. Under the assumption of small irrotational motion, the velocity potential ϕ satisfies along the free boundary the condition‡

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial n} = 0. \tag{2}$$

For simple harmonic motions with time entering as $e^{i\sigma t}$, this reduces to (1) with

$$a = -\sigma^2/g. \tag{3}$$

Again, equation (1) may be applied, for two-dimensional motions, to the flow function ψ which is the conjugate harmonic to ϕ by assuming

* Fitting, loc. cit.

† I, this Bulletin, vol. 43 (1936), p. 873; II, this Bulletin, vol. 44 (1938), p. 443.

‡ Lamb, *Hydrodynamics*, Cambridge, 1924, p. 342.

§ Lamb, loc. cit., p. 342.

a wave propagating without change of shape in the direction of x with velocity c. Replacing $\partial^2/\partial t^2$ in (2) by $c^2\partial^2/\partial x^2$, introducing ψ, and integrating over the free boundary, one proves that (1) applies to ψ with

$$a = -g/c^2. \tag{4}$$

It will be noted that both in equation (3) and in equation (4) a is negative.

Turning to the "Green's functions" of I, we interpret the one of §§2–4 as the velocity potential due to a pulsating point source; the first two functions described in §5 would represent the velocity potential due to a small sphere which is oscillating parallel and normally to the free surface; while the two-dimensional Green's function of §6 could be used either for a pulsating line source or a line source moving with uniform velocity. In all cases the singularities are at a depth h below the free surface. The interpretation of these functions given in I for the case $R(a)>0$, if it applied now, would replace the effect of the free boundaries by proper images of the sources in an infinite fluid. However, it no longer applies, since the integrals (11), (23), $\cdots$ no longer converge. On the other hand, the representations such as (18), (23), (24) still apply, provided the path of integration be properly deformed in the complex plane. Similarly, equation (25) still applies.

For a canal of finite depth H the following boundary conditions apply at the bottom $y=0$:

$$\psi = 0, \tag{5}$$

$$\frac{\partial \phi}{\partial n} = 0. \tag{6}$$

The last boundary condition along with the boundary condition (1) I investigated in II, §3, once more as regards the "reflection" of singularities, and again confining myself to the case $R(a)>0$. Dr. Weinstein has considered the boundary conditions of the form (1) and (5) along the two parallel boundaries of a strip and for negative a.* He showed that except possibly for the functions

$$\sinh \alpha y \sin \alpha x, \quad \sinh \alpha y \cos \alpha x \tag{7}$$

there exist no bounded non-null solutions of the problem. Obviously (7) will satisfy (1) at $y=H$ provided

* Rendiconti Lincei, 1927, p. 259; Comptes Rendus de l'Académie des Sciences, Paris, 1927, p. 479; Die Naturwissenschaften, 1929, p. 381.

$$\alpha \coth \alpha H + a = 0.^{*} \tag{8}$$

A different treatment of the same subject was given by Hoheisel† who first reflects across the boundary $y=0$ then reflects the function $\partial u/\partial y + au$ across the other boundary $y=H$ in a manner entirely similar to the one employed in I, §2, II, §3. Hoheisel is led to a difference-differential equation for the harmonic function involving its values and its y derivatives at (x, y) and $(x, y+2H)$.

The above is of interest in connection with the results of I, II in pointing out further the difference that might exist between a case where only the real part of a is positive and when it is negative. The existence of harmonic solutions of the proper homogeneous boundary conditions would render the corresponding Green's functions non-unique to within an additive linear combination of these solutions. From Weinstein's results it follows, therefore, that the Green's function of II, §3, if existent for $a<0$ and bounded at infinity, is also unique. The solutions (7), (8) of Weinstein are the residues of the integrands of the integrals of the form II, (17).

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* The surface waves represented by the above were investigated first by Airy (see Lamb, loc. cit., §§227–229). Equation (5) of §229 takes the place of the above (8) due to the fact that the velocity potential $\phi = \cosh \alpha y \sin \alpha x$ rather than the flux function (7) is employed.

† Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 39 (1930), p. 54. For this reference I am also indebted to Dr. Weinstein.

POLYGENIC FUNCTIONS WHOSE ASSOCIATED ELEMENT-TO-POINT TRANSFORMATION CONVERTS UNIONS INTO POINTS*

EDWARD KASNER

1. **Introduction.** A function $w=\phi(x, y)+i\psi(x, y)$ is called a *polygenic function* of the complex variable $z=x+iy$ if the real functions ϕ and ψ are general, that is, are not required to satisfy the Cauchy-Riemann equations. The value of the derivative of a polygenic function at a point z_0 depends in general not only on the point z_0 but also on the direction θ along which z approaches z_0; that is, dw/dz is of the form $F(x, y, \theta)$. Thus the derivative $\gamma=dw/dz$ of a polygenic function may be regarded as determining a correspondence between the lineal elements (x, y, θ) of the z-plane and the points (α, β) of the γ-plane, where $\gamma=\alpha+i\beta$. We call this correspondence the *element-to-point transformation T associated with the polygenic function w.*

In previous papers (Kasner, *A new theory of polygenic functions*, Science, vol. 66 (1927), pp. 581–582; *General theory of polygenic functions*, Proceedings of the National Academy of Sciences, vol. 13 (1928), pp. 75–82; *The second derivative of a polygenic function*, Transactions of this Society, vol. 30 (1928), pp. 805–818) we have shown that the element-to-point transformation T associated with a polygenic function must possess the two following properties:

I. *Elements at a given point in the z-plane correspond to points of a circle I in the γ-plane.*

II. *Corresponding central angles of the circle and angles at the point are in the ratio $-2:1$.*

If an element-to-point transformation T possesses the property I, then we define the function $H+iK$, which as a vector represents the center of the circle I, to be the *center function* of T, and the function $(H+h)+i(K+k)$, which as a vector represents the point (called the *initial point* of the circle) on the circle I which corresponds to the initial direction $\theta=0$ in the z-plane, we define to be the *principal phase function* of T. The circle I together with its initial point we call a *clock*.

We then find (Kasner, *A complete characterization of polygenic functions*, Proceedings of the National Academy of Sciences, vol. 22

* Presented to the Society, September 6, 1938.

(1936), pp. 172–177), that the element-to-point transformation T associated with a polygenic function possesses the following additional property:

III. *The principal phase point of the clock representing the derivative of the center function of T coincides with the center of the clock representing the derivative of the principal phase function of T.*

In the paper last cited, it is also proved that *for an element-to-point transformation to be associated with a polygenic function, it is necessary and sufficient that it possess the properties* I, II, *and* III.

The associated transformation T carries a single element into a point, and it carries the ∞^1 elements at a point in the z-plane into the points of a circle in the γ-plane (property I). However, a given point in the γ-plane will correspond, in general, not to a single element in the z-plane, but to a series (∞^1 elements). Now we inquire *under what conditions will this series be a union (curve or point).* Of course we mean that this shall happen for all the points of the γ-plane, that is, we demand that *all* the series so formed shall be unions. It turns out analytically that this problem means that a certain pair of functions of (x, y, p) shall be *in involution.* In our discussion, we do not demand that the jacobians be different from zero; therefore our solution will include degenerate cases. But actually the major part of the solution is not degenerate.

Our problem is thus to determine a certain specific class of polygenic functions, namely, that class for which, instead of associated series, we obtain unions. *This class, we find by a long analytic discussion, consists of the following three distinct types:*

(A) *The monogenic functions $w=f(z)$.*

(B) *The mixed quadratic fractional polygenic functions*

$$w = -\frac{az+b}{\bar{a}(\bar{a}\bar{z}+\bar{b})} + cz + d, \qquad a \neq 0.$$

(C) *The affine linear polygenic functions $w=Az+B\bar{z}+C$, $(B\neq 0)$.*

Of these three types, the quadratic type (B) is the essentially significant result revealed by our investigation.

2. **The associated element-to-point transformation T of a polygenic function.** Let the element-to-point transformation T

$$\gamma = \alpha(x, y, \theta) + i\beta(x, y, \theta) \tag{1}$$

possess the properties I, II, and III. Then we find that T can be written in the form

$$\alpha = H + h\cos 2\theta + k\sin 2\theta, \qquad \beta = K - h\sin 2\theta + k\cos 2\theta, \tag{2}$$

where H, K, h, k are functions of x and y only which satisfy

$$H_x - K_y = h_x + k_y, \quad K_x + H_y = k_x - h_y. \tag{3}$$

Let $w=\phi(x, y)+i\psi(x, y)$ be any polygenic function to which T is the associated element-to-point transformation. Then w must satisfy the two equations

$$\frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right]w = H + iK, \quad \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]w = h + ik. \tag{4}$$

From (4) it can very easily be proved that *any two polygenic functions which have the same associated element-to-point transformation T differ merely by a complex constant.*

3. **Polygenic functions whose associated element-to-point transformations convert unions into points.** We prove the following theorem:

THEOREM. *The totality of polygenic functions whose associated element-to-point transformations convert unions of the z-plane into the points of the γ-plane consists of the three distinct types* (A), (B), (C), *as specified at the end of* §1.

The proof will occupy the next three pages. Upon writing $p=\tan\theta$ the equations (2) become

$$\alpha = \frac{(H + h) + 2kp + (H - h)p^2}{1 + p^2}, \qquad \beta = \frac{(K + k) - 2hp + (K - k)p^2}{1 + p^2}. \tag{5}$$

First let us consider the case in which h and k are both zero. Then from (5) we see that our element-to-point transformation becomes a point-to-point transformation. Hence when $h=k=0$, the points of the z-plane become the points of the γ-plane, and the condition of our theorem is therefore satisfied. From (4) we find that w must be a monogenic function of z. Henceforth we shall suppose that at least one of the functions h and k is different from zero.

For the element-to-point transformation (5) to convert unions of the z-plane into the points of the γ-plane it is necessary and sufficient that

$$\frac{\beta_x + p\beta_y}{\alpha_x + p\alpha_y} = \frac{\beta_p}{\alpha_p}; \tag{6}$$

that is, the functions α and β must be in involution. Substituting (5) into (6) and making use of the equations (3), we obtain

$$\frac{(K_x + k_x) + p(H_x - 3h_x) + p^2(-H_y - 3h_y) + p^3(K_y - k_y)}{(H_x + h_x) + p(-K_x + 3k_x) + p^2(K_y + 3k_y) + p^3(H_y - h_y)} = \frac{h + 2kp - hp^2}{-k + 2hp + kp^2}. \tag{7}$$

Since the equation (7) is an identity in p, we obtain, upon setting the coefficients of the powers of p equal to zero and making use of the equations (3), the equations

$$\frac{h}{k} = \frac{-K_x - k_x}{H_x + h_x} = \frac{3H_x - h_x}{3K_x - k_x} = \frac{H_y + k_x}{K_y - h_x} = \frac{H_x + k_y}{K_x - h_y} = \frac{3H_y + h_y}{3K_y + k_y} = \frac{K_y - k_y}{-H_y + h_y}. \tag{8}$$

From the equations (8) it follows by ratio and proportion that

$$\frac{h}{k} = \frac{H_y - K_x}{H_x + K_y} = \frac{H_x + K_y}{-H_y + K_x}. \tag{9}$$

Since all the functions are real, it follows from (9) that

$$H_x = -K_y, \qquad H_y = K_x. \tag{10}$$

From (10) we find that $H+iK$ is an analytic function of $\bar{z}$; that is

$$H + iK = \lambda(\bar{z}), \tag{11}$$

where $\lambda(\bar{z})$ is an analytic function of $\bar{z}$.

Substituting $K_x = H_y$ and $K_y = -H_x$ into (8), we find that the equations (8) become

$$\frac{h}{k} = \frac{-H_y - k_x}{H_x + h_x} = \frac{3H_x - h_x}{3H_y - k_x} = \frac{H_x + k_y}{H_y - h_y} = \frac{3H_y + h_y}{-3H_x + k_y}. \tag{12}$$

Also substituting $K_x = H_y$ and $K_y = -H_x$ into equations (3), we obtain

$$2H_x = h_x + k_y, \qquad 2H_y = -h_y + k_x. \tag{13}$$

Upon substituting (13) into (12), we find

$$\frac{h}{k} = \frac{h_y - 3k_x}{3h_x + k_y} = \frac{h_x + 3k_y}{-3h_y + k_x}. \tag{14}$$

These two equations are equivalent to the equations

$$\begin{aligned} 3hh_x + 3kk_x &= kh_y - hk_y, \\ 3hh_y + 3kk_y &= hk_x - kh_x. \end{aligned} \tag{15}$$

The equations (15) are then equivalent to the equations

$$\begin{aligned} \frac{3}{2}\frac{\partial}{\partial x}\log(h^2 + k^2) &= -\frac{\partial}{\partial y}\arctan k/h, \\ \frac{3}{2}\frac{\partial}{\partial y}\log(h^2 + k^2) &= \frac{\partial}{\partial x}\arctan k/h. \end{aligned} \tag{16}$$

From (16), it follows that $(3/2)\log(h^2+k^2)+i\arctan k/h$ is an analytic function of $\bar{z}$. Thence $\exp\{(3/2)\log(h^2+k^2)+i\arctan k/h\}$ is an analytic function of $\bar{z}$; that is

$$(h^2 + k^2)(h + ik) = \mu(\bar{z}), \tag{17}$$

where $\mu(\bar{z})$ is an analytic function of $\bar{z}$. Moreover $\mu(\bar{z})\neq 0$, since at least one of the quantities h, k is different from zero.

Now (17) may be written in the form

$$(h - ik)(h + ik)^2 = \mu(\bar{z}). \tag{18}$$

Upon taking the conjugate of the equation (18), we obtain

$$(h + ik)(h - ik)^2 = \bar{\mu}(z). \tag{19}$$

Solving the equations (18) and (19) for $h+ik$, we find

$$h + ik = [\mu(\bar{z})]^{2/3}/[\bar{\mu}(z)]^{1/3}. \tag{20}$$

It is seen that the condition of our theorem is satisfied if the four functions H, K, h, k satisfy the equations (3), (11), and (20). From these equations, we find that w must be an analytic polygenic function of x and y. Hence w may be written as an analytic function of z and $\bar{z}$; that is

$$w = f(z, \bar{z}), \tag{21}$$

where f is an analytic function of z and $\bar{z}$.

From equations (4), (11), (20), and (21), we find that

$$f_z = \lambda(\bar{z}), \qquad f_{\bar{z}} = [\mu(\bar{z})]^{2/3}/[\bar{\mu}(z)]^{1/3}. \tag{22}$$

The equations (3) are then equivalent to

$$\frac{\lambda'(\bar{z})}{[\mu(\bar{z})]^{2/3}} = \frac{d}{dz}[\bar{\mu}(z)]^{-1/3}. \tag{23}$$

From (23) we find that

$$\frac{\lambda'(\bar{z})}{[\mu(\bar{z})]^{2/3}} = a, \qquad \frac{d}{dz}[\bar{\mu}(z)]^{-1/3} = a, \tag{24}$$

where a is a complex constant. From (24) we obtain

$$\lambda'(\bar{z}) = \frac{a}{(\bar{a}\bar{z}+\bar{b})^2}, \qquad \mu(\bar{z}) = \frac{1}{(\bar{a}\bar{z}+\bar{b})^3}. \tag{25}$$

First let us suppose that a is zero. Then from (22) and (25) we find that f_z and $f_{\bar{z}}$ are both constants. Thus w is the affine linear polygenic function $w=Az+B\bar{z}+C$ where $B\neq 0$.

Next let $a\neq 0$. Then from (22) and (25) we find that

$$f_z = -\frac{a}{\bar{a}(\bar{a}\bar{z}+\bar{b})} + c, \qquad f_{\bar{z}} = \frac{az+b}{(\bar{a}\bar{z}+\bar{b})^2}. \tag{26}$$

From (26) we see that our polygenic function w must be the mixed quadratic fractional polygenic function

$$w = -\frac{az+b}{\bar{a}(\bar{a}\bar{z}+\bar{b})} + cz + d, \tag{27}$$

where $a\neq 0$, b, c, d are complex constants. This completes the proof.

4. **The unions which under the associated element-to-point transformation become points.** We consider the three classes of functions mentioned in the theorem.

(A) *The monogenic functions $w=f(z)$.* Let the monogenic function $w=f(z)$ be not an affine linear monogenic function. Then the elements at any point z of the z-plane are converted into a point γ of the γ-plane and conversely. *Thus, for a monogenic function which is not affine linear, the ∞^2 point-unions of the z-plane are converted into the ∞^2 points of the γ-plane, and conversely.*

On the other hand, let w be an affine linear monogenic function. Then the derivative of w is constant; hence in the γ-plane we have a single fixed point. To this fixed point corresponds the *opulence* (the totality of ∞^3 elements) of the z-plane. *Thus for an affine linear monogenic function, the opulence of the z-plane is converted into a fixed point of the γ-plane.*

In the geometry of lineal elements of the plane, a set of ∞^1 elements is called a *series*, a set of ∞^2 elements is called a *field*, and the totality of ∞^3 elements is called the *opulence*.

(B) *The mixed quadratic fractional polygenic functions*

$$w = -\frac{az+b}{\bar{a}(\bar{a}\bar{z}+\bar{b})} + cz + d.$$

The unions in the z-plane, which under the associated element-to-point transformation of the polygenic function w become the points of the γ-plane, are *the ∞^2 circles through the point $-b/a$ and the field defined by the ∞^1 straight lines through the point $-b/a$.*

To a point $\gamma \neq c$ of the γ-plane there corresponds a definite circle of the z-plane through the point $-b/a$, and conversely. The center C and the radius R are given by the formulas

$$C = -\frac{b}{a} + \frac{\bar{a}}{a^2(\bar{c}-\bar{\gamma})}, \qquad R^2 = \frac{1}{a\bar{a}(c-\gamma)(\bar{c}-\bar{\gamma})}.$$

The field defined by the pencil of straight lines through the point $-b/a$ of the z-plane is converted into the point c of the γ-plane.

(C). *The affine linear polygenic functions* $w = Az + B\bar{z} + C$, $(B \neq 0)$. The associated element-to-point transformation of the affine linear polygenic function $w = Az + B\bar{z} + C$, $(B \neq 0)$, converts the *opulence* (the totality of ∞^3 elements) of the z-plane into the ∞^1 points of the circle in the γ-plane whose center is A and whose radius is $|B| \neq 0$.

It is found that *to any point of the fixed circle in the γ-plane, there corresponds the field defined by ∞^1 parallel straight lines, and conversely.*

5. **Scholium.** We thus find that there are four distinct geometric possibilities in the z-plane:

(A′). The ∞^2 point-unions (stars).

(A″). The *opulence* of elements in the z-plane.

(B). The ∞^2 circles through a fixed point together with the field defined by the pencil of straight lines through the same fixed point.

(C). The ∞^1 fields defined by parallel straight lines.

In the γ-plane, we find the following three distinct geometric possibilities:

(A′, B). The ∞^2 points.

(C). The ∞^1 points of a fixed circle.

(A″) A single fixed point.

We remark in conclusion that the quadratic type (B), formula (27), gives the really significant configuration.

COLUMBIA UNIVERSITY

ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES*

W. C. RANDELS

1. **Introduction.** A series $\sum u_n$ is said to be absolutely summable by a method $\mathfrak{a}$ defined by a matrix a_{mn} if

$$\sum_{m=1}^{\infty} \left| S_m(\mathfrak{a}, u) - S_{m-1}(\mathfrak{a}, u) \right| < \infty ,$$

where

$$S_m(\mathfrak{a}, u) = \sum_{n=0}^{\infty} a_{mn} u_n .$$

Similarly a series is said to be absolutely summable $|A|$ if

$$S(r, u) = \sum_{n=0}^{\infty} u_n r^n \subset BV \quad \text{on} \quad (0, 1).$$

It is known that if $\sum u_n$ is absolutely summable $|C_\alpha|$ for some $\alpha > 0$, then it is absolutely summable $|A|$. There are, however, series absolutely summable $|A|$ but not $|C_\alpha|$ for any α whatever. We intend to give here an example of a Fourier series with that property.

Bosanquet† has proved that, if the Fourier series of $f(x)$ is absolutely summable $|C_\alpha|$, then the function

$$\phi_\beta(f, t) = \beta t^{-\beta} \int_0^t \{f(x+\tau) + f(x-\tau) - 2f(x)\} \, (t-\tau)^{\beta-1} d\tau$$

is of bounded variation on $(0, \pi)$ for $\beta > \alpha$; and conversely, if $\phi_\alpha(t)$ is of bounded variation, the Fourier series of $f(x)$ is absolutely summable $|C_\beta|$, $(\beta > \alpha + 1)$.

2. **Preliminary definitions.** Let α_{nk}, β_{nk} be defined for $n = 1, 2, \cdots$, $k = 1, 2, \cdots, n$, by

$$\alpha_{nk} = 2^{-k-n-n/(k-1/2)}, \qquad \beta_{nk} = 2^{-n} - 2^{-n-n/(k-1/2)}. \tag{1}$$

Then, since $k \leqq n$, we have

$$\beta_{nk} > 2^{-n-1}.$$

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* Presented to the Society, December 30, 1937.

† L. S. Bosanquet, *The absolute Cesàro summability of Fourier series*, Proceedings of the London Mathematical Society, vol. 41 (1936), pp. 517–528.

Let $f_{nk}(x)$ be defined over $-\pi \leqq x \leqq \pi$, so that

(2) $f_{nk}(x) = 2^n, \quad \beta_{nk} \leqq |x| \leqq \beta_{nk} + \alpha_{nk},$

(3) $f_{nk}(x) = -f_{nk}(x - 2^j\alpha_{nk}), \quad \beta_{nk} + 2^j\alpha_{nk} < |x| \leqq \beta_{nk} + 2^{j+1}\alpha_{nk},$

(4) $f_{nk}(x) = 0$ elsewhere on $(-\pi, \pi)$,

where in (3) j takes on the values $0, \cdots, k-1$. The relation (3) implies that

$$\int_{\beta_{nk}}^{\beta_{nk}+2\alpha_{nk}} f_{nk}(x)dx = \int_{\beta_{nk}}^{\beta_{nk}+\alpha_{nk}} f_{nk}(x)dx - \int_{\beta_{nk}}^{\beta_{nk}+\alpha_{nk}} f_{nk}(x)dx = 0,$$

and by induction

$$\int_{\beta_{nk}}^{\beta_{nk}+2^j\alpha_{nk}} f_{nk}(x)dx = 0, \qquad 1 \leqq j \leqq k.$$

If we define

$$\Phi_1(f, t) = \int_0^t \{f(x) + f(-x) - 2f(0)\}dx,$$

we can obtain the following relations analogous to (3) and (4):

(5) $\Phi_1(f_{nk}, t) = -\Phi_1(f_{nk}, t - 2^{j+1}\alpha_{nk}), \quad \beta_{nk} + 2^{j+1}\alpha_{nk} < x \leqq \beta_{nk} + 2^{j+2}\alpha_{nk},$

(6) $\Phi_1(f_{nk}, t) = 0$ elsewhere on $(0, \pi)$,

where in (5) j takes on the values $0, \cdots, k-2$.

We define by induction the functions

$$\Phi_{r+1}(f, t) = (r+1)\int_0^t \Phi_r(f, x)dx,$$

for which it can be shown by similar reasoning that, for $r \leqq k$,

(7) $\Phi_r(f_{nk}, t) = -\Phi_r(f_{nk}, t - 2^{r+j}\alpha_{nk}), \quad \beta_{nk} + 2^{r+j}\alpha_{nk} < x \leqq \beta_{nk} + 2^{r+j+1}\alpha_{nk},$

(8) $\Phi_r(f_{nk}, t) = 0,$ elsewhere on $(0, \pi)$,

where in (7) j takes on the values $0, \cdots, k-r-1$. We notice that, at $x = 0$, $\phi_r(f, t) = t^{-r}\Phi_r(f, t)$, and therefore for $r \leqq k-1$

$$\begin{aligned}\phi_r(f_{nk}, \beta_{nk} + \alpha_{nk}) &= 2r(\beta_{nk} + \alpha_{nk})^{-r}\int_{\beta_{nk}}^{\beta_{nk}+\alpha_{nk}} 2^n(\beta_{nk} + \alpha_{nk} - x)^{r-1}dx \\ &= 2^{n+1}r(\beta_{nk} + \alpha_{nk})^{-r}\int_0^{\alpha_{nk}} (\alpha_{nk} - x)^{r-1}dx \\ &> 2^{n+1}r2^{nr}\alpha_{nk}^r > 2^{-kr}2^{n/2^k},\end{aligned}$$

since

$$n(r+1) - r\{n + n/(k-1/2)\} = n\{1 - r/(k-1/2)\} > n/2k.$$

This shows that, for $r<k$,

$$T.V._{(0,\pi)}\phi_r(f_{nk}, x) > 2^{-kr}2^{n/2k}.$$

On the other hand,

$$\phi_k'(f_{nk}, t) = kt^{-k}\Phi_{k-1}(f_{nk}, t) - kt^{-k-1}\Phi_k(f_{nk}, t),$$

so that, if $I=(\beta_{nk}, \beta_{nk}+2^k\alpha_{nk})=(\beta_{nk}, 2^{-n})$, then

$$\begin{aligned}\int_I |\phi_k'(f_{nk}, t)|\,dt &= O\left\{2^{kn}\int_0^{2^k\alpha_{nk}} 2^n(2^k\alpha_{nk} - t)^{k-1}dt\right.\\ &\qquad \left. + 2^{(k+1)n}\int_0^{2^k\alpha_{nk}} 2^n(2^k\alpha_{nk} - t)^k dt\right\}\\ &= O(2^{-n/2k}).\end{aligned}$$

Therefore

$$T.V._{(0,\pi)}\phi_k(f_{nk}, t) = O(2^{-n/2k}).$$

3. **Definition of $f(x)$.** We define the functions

$$f_k(x) = \sum_{[\log_2 k]+1}^{\infty} f_{2^n+k,k}(x).$$

For $r \leqq k$

$$\phi_r(f_{2^n+k,k}, t)\cdot\phi_r(f_{2^m+k,k}, t) = 0, \qquad m \neq n,$$

and therefore

$$T.V._{(0,\pi)}\phi_r(f_k, t) = \sum_{[\log_2 k]+1}^{\infty} T.V._{(0,\pi)}\phi_r(f_{2^n+k,k}, t) = \infty, \quad r < k,$$

and

$$\begin{aligned}(9)\qquad T.V._{(0,\pi)}\phi_k(f_k, t) &= \sum_{[\log_2 k]+1}^{\infty} T.V._{(0,\pi)}\phi_k(f_{2^n+k,k}, t)\\ &= O\left(\sum_1^{\infty} 2^{-i/2k}\right) = O(1).\end{aligned}$$

It follows then that, for $s>k$,

$$\phi_s(f_k, t) \subset BV \quad \text{on} \quad (0, \pi).$$

The Fourier series of $f_k(x)$ must be absolutely summable $|A|$, at $x=0$. We set

$$A_k = T.V._{(0,1)} \frac{1}{\pi} \int_0^{\pi} f_k(t) \frac{1-r^2}{1-2r\cos t + r^2}\, dt.$$

A sequence d_k is then defined so that

$$d_k \leqq A_k 2^{-k}, \tag{10}$$

$$d_k \leqq 2^{-k} \int_0^{\pi} |f_k(x)|\, dx. \tag{11}$$

The function

$$f(x) = \sum_1^{\infty} d_k f_k(x)$$

is the one we set out to construct. By (11), $f(x) \subset L$, since

$$\int_0^{\pi} |f(x)|\, dx \leqq \sum_1^{\infty} d_k \int_0^{\pi} |f_k(x)|\, dx \leqq \sum_1^{\infty} 2^{-k} = 1.$$

We have, by (10),

$$T.V._{(0,1)} \frac{1}{\pi} \int_0^{\pi} f(t) \frac{1-r^2}{1-2r\cos t + r^2}\, dt \leqq \sum_1^{\infty} d_k A_k \leqq \sum_1^{\infty} 2^{-k} = 1,$$

which means that the Fourier series of $f(x)$ is absolutely summable $|A|$, at $x=0$. Finally, using (9) we see that

$$T.V._{(0,\pi)}\phi_j(f, t) > T.V._{(0,\pi)}\phi_j(f_j, t) - \left| \sum_1^{j-1} T.V._{(0,\pi)}\phi_j(f_k, t) \right|$$

$$= \infty - O(1) = \infty;$$

so the Fourier series of $f(x)$ cannot be $|C_j|$ summable at $x=0$, for any j. This completes the proof of our assertion.

NORTHWESTERN UNIVERSITY

ALGEBRAIC POSTULATES AND A GEOMETRIC INTERPRETATION FOR THE LEWIS CALCULUS OF STRICT IMPLICATION

TANG TSAO-CHEN

1. **Two further postulates for a Boolean ring with a unit element.** If addition, subtraction, and multiplication are properly defined in logic, it may be shown* that the postulates for these operations are identical with those in a ring, in which every element is idempotent, satisfying the postulate $xx=x$. Such a ring is called a Boolean ring. The postulates for a Boolean ring with a unit element are therefore the following:

A. *Addition is always possible, commutative, and associative.*

B. *Multiplication is always possible, associative, and both left- and right-distributive with respect to addition.*

C. *Subtraction is always possible.*

D. $xx=x$.

E. *There exists an element* 1 *such that* $x1=x$ *for every element* x *in the ring.*

Here we shall introduce a new operation, represented by x^∞, which satisfies the following two further postulates:

F_1. *For every element* x *there exists an element* x^∞ *such that* $x^\infty x=x^\infty$.

F_2. *For any two elements* x *and* y *we have* $(xy)^\infty=x^\infty y^\infty$.

The postulates A-F_2, obtained above, may be called the algebraic postulates for the Lewis calculus of strict implication.

2. **A geometric meaning of the symbol x^∞.** A geometric meaning† may be attached to x^∞ as follows: Let x be a point set in the euclidean

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* See M. H. Stone. *The theory of representations for Boolean algebras*, Transactions of this Society, vol. 40 (1936), pp. 37–53.

† Another geometric meaning of x^∞ may be obtained by assuming 1^∞ to be any one fixed point or any set of fixed points (finite or infinite in number and continuous or discontinuous in character) and setting $x^\infty=x1^\infty$. If we assume that 1^∞ is a fixed point, we have then the following property:

G. x^∞ *is two-valued, that is,* $x^\infty=1^\infty$ or 0^∞,

which is independent of the postulates A-F_2. This sub-Boolean algebra with the postulates A-G does not become the ordinary two-valued Boolean algebra, unless we assume further that x is two-valued.

plane; then x^∞ may be taken as the interior points of x (the set of all points p such that some circle with center at p lies entirely within x). Then the postulates F_1 and F_2 are satisfied. Moreover $(1-x)^\infty$ is the set of points interior to the complement of x, that is, exterior of x; $1-x^\infty-(1-x)^\infty$ is the frontier of x; and $1-(1-x)^\infty$ is the closure of x, that is, x plus its frontier.

3. **A geometric interpretation of Lewis' possibility functions.** When p is a proposition, Lewis introduces an undefined idea $\Diamond p$, which is read "p is possible" and may be called the possibility function of p. Now, if a class x be given, $\Diamond x$ must have a corresponding meaning which is obtained from the following definition:

DEFINITION. $\Diamond x = 1-(1-x)^\infty$.

By means of the definition of $\sim x$, we obtain $\sim x = 1-x$; and we have the following theorems:

THEOREM 1. *$\Diamond x$ is the closure of x.*

THEOREM 2. *$\sim\Diamond x = (1-x)^\infty$ is the exterior of x.*

THEOREM 3. *$\Diamond\sim x = 1-x^\infty$ is the closure of the complement of x.*

THEOREM 4. *$\sim\Diamond\sim x = x^\infty$ is the interior of x.*

On the basis of the postulates A-F we may then prove abstractly the following theorems:

THEOREM 5. *$0^\infty = 0$ and $\sim\Diamond\sim 0 . = .0$; that is, the interior of the null class is a null class.*

THEOREM 6. *$\sim\Diamond\sim x . \sim\Diamond x : = .0$, and $x^\infty(\sim x)^\infty = 0$; that is, if x is a class, then the interior and the exterior of x have no point in common.*

DEFINITION. *$x^f = 1-x^\infty-(\sim x)^\infty$, that is, x^f is the frontier of x.*

THEOREM 7. *$x^f = (\sim x)^f$, that is, x and $\sim x$ have the same frontier.*

THEOREM 8. *$x^f + x^\infty + (\sim x)^\infty = 1$, $x^f + (\sim\Diamond\sim x) + (\sim\Diamond x) = 1$; that is, the frontier, the interior, and the exterior of x form the whole plane.*

THEOREM 9. *$\Diamond x . = . x^f + x^\infty$ and $\Diamond x . = . x^f + (\sim\Diamond\sim x)$; that is, $\Diamond x$ is the sum of the frontier and the interior of x.*

THEOREM 10. *$\Diamond\sim x . = . x^f + (\sim x)^\infty$ and $\Diamond\sim x . = . x^f + \sim\Diamond x$; that is, $\Diamond\sim x$ is the sum of the frontier and the exterior of x.*

THEOREM 11. *$x(\sim x)^\infty = 0$ and $x . \sim\Diamond x : = .0$; that is, the class x and its exterior have no point in common.*

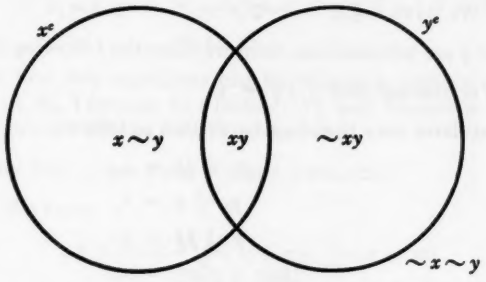
THEOREM 12. *$x = x^{\infty} + xx'$ and $x = (\sim\Diamond\sim x) + xx'$; that is, the class x is the sum of its interior and the part of its frontier which belongs to it.*

THEOREM 13. *$(\sim x) = (\sim x)^{\infty} + (\sim x)x'$ and $(\sim x) = (\sim\Diamond x) + (\sim x)x'$; that is, the class $\sim x$ is the sum of the exterior of x and the part of its frontier which belongs to $\sim x$.*

4. **Theorems corresponding to Lewis' postulates.** In *Symbolic Logic* by Lewis and Langford the symbol $p \prec q$ is defined by $p \prec q . = . \sim\Diamond(p\sim q)$. Changing the propositions p and q into the classes x and y, we get the definition of $x \prec y$:

DEFINITION. $x \prec y . = . \sim\Diamond(x\sim y)$.

The geometric meaning of $x \prec y$ is very clear, for its definition states that $x \prec y$ means the exterior of $x\sim y$; that is, if, for example, two curves x^c and y^c divide the plane into four parts xy, $\sim xy$, $x\sim y$, and $\sim x\sim y$, as in the figure, then $x \prec y$ means the interior of $x \supset y$, which is the sum of xy, $\sim xy$, and $\sim x\sim y$. In other words, we have the following two theorems:



THEOREM 14. $x \supset y . = . xy + (\sim xy) + (\sim x\sim y)$.*

THEOREM 15. $x \prec y . = . \sim\Diamond\sim(x \supset y) . = . (x \supset y)^{\infty}$.

PROOF. By the definition of $\sim x$, $\sim x . = . 1 - x$, and we see that

$$\begin{aligned} xy + (\sim xy) + (\sim x \sim y) &= xy + (1 - x)y + (1 - x)(1 - y) \\ &= xy + y - xy + 1 - x - y + xy \\ &= 1 - x + xy \\ &= (x \supset y). \end{aligned}$$

* $x \supset y$ is, of course, defined as $\sim(x\sim y)$ which is $1 - x + xy$.

By Theorem 4

$$\sim\Diamond\sim(x\supset y).=.(x\supset y)^{\infty},$$

$$\begin{aligned}\sim\Diamond\sim(x\supset y).&=.\sim\Diamond\sim(1-x+xy)\\&=.\sim\Diamond[1-(1-x+xy)]\\&=.\sim\Diamond(x-xy)\\&=.\sim\Diamond x(1-y)\\&=.\sim\Diamond(x\sim y)\\&=.x\dashv y.\end{aligned}$$

DEFINITION. $i=(\sim\Diamond 0)$.

This definition of i may be replaced by the following theorem:

THEOREM 16. *$i=1^{\infty}$ and $i=(\sim\Diamond\sim 1)$.*

PROOF. We have $i=(\sim\Diamond 0)=[\sim\Diamond(1-1)]=(\sim\Diamond\sim 1)=1^{\infty}$, by Theorem 3.

THEOREM 17. $x\dashv x.=.i$.

PROOF. We have $x\dashv x.=.\sim\Diamond(x\sim x)=.\sim\Diamond 0=.i$.

If p and q are propositions, then we have the following theorem:*

If $p\dashv q$ is asserted, then $p\dashv q.=.i$.

Lewis' postulates may therefore be written as follows:

$$\begin{aligned}pq\dashv qp.&=.i,\\pq\dashv p.&=.i,\\p\dashv pp.&=.i,\\(pq)r\dashv p(qr).&=.i,\\p\dashv\sim(\sim p).&=.i,\\p\dashv q.q\dashv r:\dashv.p\dashv r:&=.i,\\p.p\dashv q:\dashv.q:.&=.i,\\\Diamond(pq)\dashv\Diamond p.&=.i.\end{aligned}$$

The corresponding theorems for classes are obtained by changing the propositions p, q, r into x, y, z, respectively:

THEOREM 18. $xy\dashv yx.=.i$.

PROOF. We have $xy\dashv yx.=.xy\dashv xy=.i$ by Theorem 17.

* See this Bulletin, vol. 42 (1936), p. 708, Theorem 9.

THEOREM 19. $xy \dashv x . = .i.$

PROOF. We have $xy \dashv x . = . \sim\Diamond(xy\sim x) = . \sim\Diamond(y0) = . \sim\Diamond\, 0 = .i.$

In like manner we may prove Theorems 20–22 below:

THEOREM 20. $x \dashv xx . = .i.$

THEOREM 21. $(xy)z \dashv x(yz) . = .i.$

THEOREM 22. $x \dashv \sim(\sim x) . = .i.$

THEOREM 23. $x \dashv y . y \dashv z : \dashv . x \dashv z : . = .i.$

PROOF. We have

$$\begin{aligned} x \dashv y . y \dashv z : \dashv . x \dashv z : . &= : (x \supset y)^{\infty} (y \supset z)^{\infty} . \dashv . x \dashv z \\ &= : (x \supset y . y \supset z)^{\infty} . \dashv . x \dashv z \\ &= : (x \supset y . y \supset z . x \supset z)^{\infty} . \dashv . x \dashv z \\ &= : (x \supset y)^{\infty} (y \supset z)^{\infty} (x \supset z)^{\infty} . \dashv . x \dashv z \\ &= : . x \dashv y . y \dashv z . x \dashv z : \dashv . x \dashv z \\ &= : . x \dashv z . x \dashv y . y \dashv z . : \dashv . x \dashv z \\ &= .i, \end{aligned}$$

where the first five equations can be obtained, respectively, from Theorem 15, F_2, Theorem 23.1 (below), F_2, and Theorems 4 and 15. The last equation follows from Theorem 19.

THEOREM 23.1. $x \supset y . y \supset z : = : x \supset y . y \supset z . x \supset z.$

PROOF. We have

$$\begin{aligned} x \supset y . y \supset z : &= : (\sim x + xy) . (\sim y + yz) \\ &= : \sim x \sim y + \sim xyz + xy \sim y + xyyz \\ &= : \sim x \sim y + \sim xyz + 0 + xyz \\ &= : \sim x \sim y + yz. \end{aligned} \tag{1}$$

Then

$$\begin{aligned} x \supset y . y \supset z . x \supset z . : &= : (\sim x \sim y + yz) . (x \supset z) \\ &= : (\sim x \sim y + yz) . (\sim x + xz) \\ &= : \sim x \sim x \sim y + \sim xyz + 0 + xyzz \\ &= : \sim x \sim y + \sim xyz + xyz \\ &= : \sim x \sim y + yz \\ &= : x \supset y . y \supset z, \end{aligned}$$

by (1) above.

THEOREM 24. $x.x \supset y: \supset .y: . = .i.$

PROOF. We have

$$\begin{aligned} x.x \supset y: \supset .y: . &= .\sim \Diamond (x.x \supset y.\sim y) \\ &= .\sim \Diamond (x \sim y.x \supset y) \\ &= .\sim \Diamond (x \sim y.\sim \Diamond (x \sim y)) \\ &= .\sim \Diamond 0 \\ &= .i, \end{aligned}$$

by Theorem 11 and a definition.

THEOREM 25. $\Diamond(xy) \supset \Diamond x. = .i.$

PROOF. By the application of Theorem 25.1 (below),

$$\begin{aligned} \Diamond(xy) \supset \Diamond x. &= .\sim \Diamond [\Diamond (xy) \sim \Diamond x] \\ &= .\sim \Diamond [\Diamond(xy) \sim \Diamond x \sim \Diamond (xy)] \\ &= .\sim \Diamond (\sim \Diamond x.0) \\ &= .\sim \Diamond 0 \\ &= .i, \end{aligned}$$

by definition.

THEOREM 25.1. $\sim\Diamond x. = :\sim\Diamond x.\sim\Diamond(xy).$

PROOF. We have, using F_2,

$$\begin{aligned} \sim \Diamond x.\sim \Diamond (xy): &= .(1 - x)^{\infty}(1 - xy)^{\infty}, \\ &= .[(1 - x)(1 - xy)]^{\infty} \\ &= .(1 - x - xy + xxy)^{\infty} \\ &= .(1 - x)^{\infty} \\ &= .\sim \Diamond x, \end{aligned}$$

where the first step follows by Theorem 1 and the last by Theorem 2.

5. **Theorems corresponding to Lewis' operations.** Lewis' substitution can be performed both on propositions and on classes. We consider therefore only adjunction and inference. For this purpose, we first establish the following two theorems:

THEOREM 26. $x.x \supset y: = :xy.x \supset y.$

PROOF. By Theorem 11 we have

$$x \sim y . \sim \Diamond (x \sim y) : = . 0,$$

or

$$x(1 - y) . x \prec y : = . 0,$$

or

$$(x - xy) . x \prec y : = . 0,$$

or

$$(x . x \prec y) - (xy . x \prec y) = 0,$$

from which we get at once Theorem 26.

THEOREM 27. (1) *If* $i \prec x . = . i$, *then* $ix = i$. (2) *Conversely, if* $ix = i$, *then* $i \prec x . = . i$.

PROOF. We obtain, using Theorem 16, F_2, and Theorem 26, $ix = 1^\infty x = (1.1)^\infty x = 1^\infty 1^\infty x = iix = ixi = ix(i \prec x) = i(i \prec x) = ii = i$.

The proof of the converse theorem is the following: We have $i \prec x . = . \sim \Diamond (i \sim x) = . \sim \Diamond [i \sim (ix)] = . \sim \Diamond (i \sim i) = . \sim \Diamond\ 0 = . i$, by definition.

DEFINITION. *We write* $\vdash x$, *when and only when* $ix = i$.

Thus, we may write $\vdash x$, when $x = 1$ or $x = i$. For example, Theorem 18 is equivalent to $\vdash . xy \prec yx$.

We can now prove the theorems corresponding to adjunction and inference.

THEOREM 28. *If* $\vdash x$ *and* $\vdash y$, *then* $\vdash (xy)$.

PROOF. We have $i(xy) = iixy = ixiy = ii = i$. Hence by definition, $\vdash (xy)$.

THEOREM 29. *If* $\vdash x$, *and* $\vdash . x \prec y$, *then* $\vdash y$.

PROOF. We obtain $iy = iiy = (ix)(i . x \prec y)y = ixi(x \prec y)y = iixy(x \prec y) = iix(x \prec y) = (ix)i(x \prec y) = ii = i$, using Theorem 26. Therefore $iy = i$; hence by definition $\vdash y$.

6. **Miscellaneous theorems.** We state the following theorems, each without proof:

THEOREM 30. $x \prec y . = : i . x \prec y$.

THEOREM 31. *If* $x \prec y . = . i$, *and* $y \prec z . = . i$, *then* $x \prec z . = . i$.

THEOREM 32. *If* $x \supset y. = .i$ *and* $x \supset y. \supset .u \supset v := .i$, *then* $u \supset v. = .i$.

THEOREM 33. (1) *If* $x \supset y. = .i$, *then* $\vdash .x \supset y$. (2) *Conversely, if* $\vdash .x \supset y$, *then* $x \supset y. = .i$.

THEOREM 34. (1) *If* $i \supset x. = .i$, *then* $\vdash x$. (2) *Conversely, if* $\vdash x$, *then* $i \supset x. = .i$.

THEOREM 35. *If* $1 \supset x. = .i$, *then* $\vdash x$.

7. **A relation between Huntington's "p in T" and "$\vdash p$" in this paper.** In this Bulletin (vol. 40 (1934), p. 733) E. V. Huntington established the theorem

$$(p < q) \rightleftarrows [(p \supset q) \text{ in } T] \tag{2}$$

from the following definition of $(p<q)$:

$$(p < q) \rightleftarrows (p = pq). \tag{3}$$

It is now my aim to prove the following theorem:

THEOREM 36. $[(p \supset q) \text{ in } T] \rightleftarrows (\vdash .p \supset q)$.

PROOF OF $[(p \supset q)$ in $T] \rightarrow (\vdash .p \supset q)$. By hypothesis, $(p \supset q)$ in T. By (2) and (3), $p = pq$. Therefore, $p \supset q. = .pq \supset q. = .i$. Thus $\vdash .p \supset q$.

PROOF OF $(\vdash .p \supset q) \rightarrow [(p \supset q)$ in $T]$. By (2) and (3) $(p = pp) \rightarrow [(p \supset p)$ in $T]$. Therefore, $(p \supset p)$ in T, that is,

$$i \text{ in } T. \tag{4}$$

But by hypothesis, $i.p \supset q := .i$. Hence

$$p \supset q. = .i. \tag{5}$$

From (4) and (5) we have at once $(p \supset q)$ in T.

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